Cambridge Tracts in Mathematics
and Mathematical Physics

General Editors
J. G. LEATHEM, M.A.
E. T. WHITTAKER, M.A., F.R.S.

No. 2

THE INTEGRATION OF FUNCTIONS OF A SINGLE VARIABLE

by

G. H. HARDY, M.A.

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by

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Fellow of Trinity College

CAMBRIDGE:
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PREFACE.

THIS pamphlet is intended to be read as a supplement to the accounts of 'Indefinite Integration' given in text-books on the Integral Calculus. The student who is only familiar with the latter is apt to be under the impression that the process of integration is essentially 'tentative' in character, and that its performance depends on a large number of disconnected though ingenious devices. My object has been to do what I can to show that this impression is mistaken, by showing that the solution of any elementary problem of integration may be sought in a perfectly definite and systematic way.

The reader who is familiar with the theory of algebraical functions and algebraical plane curves will no doubt find the treatment in Section V. of the integrals of algebraical functions sketchy and inadequate. I hope, however, that he will bear in mind the great difficulty of presenting even an outline of the elements of so vast a subject in a short space and without presupposing a wider range of mathematical knowledge than I am at liberty to assume.

I have naturally not said much about particular devices which are only useful in special cases, but I have tried to show, where it is possible, how such devices find their place in the general theory. And I would strongly recommend any reader who is not already familiar with the general processes here explained to work through a number of examples (those for instance which have been set in the Mathematical Tripos in recent years) using in each case both the general method and any special method which he may find better adapted to the particular case.

I have borrowed largely from the *Cours d'Analyse* of Hermite and Goursat, but my greatest debt is to Liouville, who published in the years 1830–40 a series of remarkable memoirs on the general problem of integration which appear to have fallen into an oblivion which they certainly do not deserve. It was Liouville who first gave rigid proofs of whole series of theorems of the most fundamental importance in analysis—that the exponential function is not algebraical, that the logarithmic function cannot be expressed by means of algebraical and exponential functions, and that the standard elliptic integrals cannot be expressed by algebraical, exponential and logarithmic functions. That such theorems require proof is too often altogether forgotten.

I have added a list of references for the benefit of more advanced readers.

G. H. H.

November, 1905.

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THE INTEGRATION OF FUNCTIONS OF A SINGLE VARIABLE.

I. Introduction.

The subject of the following pages is what may fairly be described as the fundamental problem of the Integral Calculus properly so called: '*to find a function whose differential coefficient is a given function*,' or to solve the differential equation

$$\frac{dy}{dx} = f(x) \qquad (1).$$

It may seem at first sight that the Integral Calculus thus defined is merely a very small department of the theory of Differential Equations. Indeed Euler, the first systematic writer on the Calculus, defines the Integral Calculus in a way which includes the whole of that theory in its scope: 'calculus integralis est methodus, ex data differentialium relatione inveniendi relationem ipsarum quantitatum*.' Or again it may seem as if, according to our definition, the Integral Calculus is only a small part of the Theory of Definite Integrals. The latter theory, starting from the definition of the definite integral

$$\int_{t_0}^{t_1} f(t)\,dt,$$

as the limit of a certain sum, shows us that, under certain conditions on which we need not insist, the solution of the equation (1) is given by

$$y = \int_{t_0}^{x} f(t)\,dt.$$

Every problem of what is usually (though not very happily) called 'indefinite integration' may therefore be regarded as a problem in the

* *Institutiones Calculi Integralis*, p. 1.

theory of definite integrals, while the latter theory obviously includes many problems which fall outside the former.

In spite of this 'indefinite integration' in reality forms an independent theory, proceeding by its own methods and meeting with difficulties peculiar to itself. When we say that we have solved a differential equation,

$$f\left(x,\ y,\ \frac{dy}{dx}\right)=0$$

for example, we mean that we have succeeded either in expressing y explicitly in terms of x by functional signs, one of which may be the sign of indefinite integration, or implicitly by means of some relation such as an algebraical equation. We have in other words removed the difficulties of the problem from the field proper to the theory of differential equations to that of some other theory whose results are taken for granted. If our result involves the sign of indefinite integration the further question arises as to whether the process indicated can actually be carried out, and this is a question not in differential equations but in integral calculus.

Much the same may be said of the relations of the theory of 'indefinite integration' to the theory of definite integrals, or rather to the part of the latter theory which is concerned with the evaluation of particular integrals. To evaluate

$$\int_0^\infty \phi(x,\ y)\,dx,$$

for instance, is to express it explicitly as a function of y, and in this expression the sign of indefinite integration may perfectly well occur.

With the other side of the theory of definite integrals, the side which is really part of what is called the 'Theory of Functions of Real Variables,' and deals with questions concerning limits, continuity, and convergence, our present subject has really very little connection. We may indeed draw from it the one result, to which allusion has already been made, as to the existence of a theoretical solution of the equation (1). But from our present point of view this result is entirely uninteresting and unimportant. What we are concerned with is the *form* of the solution, and the only proof of its existence which is of any value to us is that which consists in actually expressing it in terms of x. And we shall not be troubled in the least by any difficulties concerning continuity. The functions with which we shall be dealing will be always such that they and their differential coefficients are continuous except for certain special values of x, and these values of x we shall simply omit from consideration. It no way affects the meaning of the equations

$$\frac{d \log x}{dx} = \frac{1}{x}, \quad \int \frac{dx}{x} = \log x$$

that $\log x$ and $1/x$ become infinite for $x = 0$.

After these preliminary remarks we may proceed to define our subject more precisely.

II. Elementary functions and their classification.

An *elementary function* is a member of the class of functions which comprises

(i) rational functions,

(ii) algebraical functions, explicit or implicit,

(iii) the exponential function e^x,

(iv) the logarithmic function $\log x$,

(v) all functions which can be defined by means of any finite combination of the symbols proper to the preceding four classes of functions.

A few remarks and examples may help to elucidate this definition.

1. A *rational function* is a function defined by means of any finite combination of the elementary operations of addition, multiplication, and division, operating on the variable x.

It is shown in elementary algebra that any rational function of x may be expressed in the form

$$f(x) = \frac{a_0 x^m + a_1 x^{m-1} + \dots + a_m}{b_0 x^n + b_1 x^{n-1} + \dots + b_n},$$

where m and n are positive integers and the a's and b's constants. It is hardly necessary to remark that it is in no way involved in the definition of a rational function that these constants should be rational or algebraical * or real *numbers*. Thus

$$\frac{x^2 + x + i\sqrt{2}}{x\sqrt{2} - e}$$

is a rational function.

* An algebraical number is a number which is the root of an algebraical equation whose coefficients are integral. It is known that many numbers (such as e and π) are not roots of any such equation.

2. An *explicit algebraical function* is a function defined by means of any finite combination of the four elementary operations and any finite number of operations of root extraction. Thus

$$\frac{\sqrt{(1+x)}-\sqrt[3]{(1-x)}}{\sqrt{(1+x)}+\sqrt[3]{(1-x)}},\quad \sqrt{\{x+\sqrt{(x+\sqrt{x})}\}},\quad \left(\frac{x^2+x+i\sqrt{2}}{x\sqrt{2}-e}\right)^{\frac{2}{3}}$$

are explicit algebraical functions. And so is $x^{\frac{m}{n}}$ (*i.e.* $\sqrt[n]{x^m}$) for any integral values of m and n. But

$$x^{\sqrt{2}},\quad x^{1+i}$$

are not algebraical functions at all, but transcendental functions, as irrational or complex powers can only be defined by the aid of exponentials and logarithms.

If y is an explicit algebraical function of x we can always find an equation

$$y^m + R_1 y^{m-1} + \dots + R_m = 0,$$

whose coefficients are rational functions of x. Thus, for example, the function

$$y = \sqrt{x} + \sqrt{(x+\sqrt{x})}$$

satisfies the equation

$$y^4 - (4y^2 + 4y + 1)\,x = 0.$$

The converse is not true, since it has been proved that in general equations of degree higher than the fourth have no roots which are explicit algebraical functions of their coefficients. A simple example is given by the equation

$$y^5 - y - x = 0.$$

We are thus led to consider a more general class of functions, *implicit* algebraical functions, which includes the class of explicit algebraical functions.

3. An *algebraical function* of x is a function which satisfies an equation

$$y^m + R_1 y^{m-1} + \dots + R_m = 0,$$

whose coefficients are rational functions of x.

We shall always suppose this equation to be *irreducible*, *i.e.* incapable of resolution into factors whose coefficients are also rational functions of x. If it could be so resolved we could regard y as the root of an equation of lower degree than m. Thus if $y^4 - x^2 = 0$ we must have either $y^2 + x = 0$ or $y^2 - x = 0$. Each of these latter equations is irreducible.

The equation which y satisfies will have $m-1$ roots other than y. No two roots can be equal, for if two roots were equal the equation would have a factor in common with the derived equation

$$my^{m-1} + (m-1) R_1 y^{m-2} + \ldots = 0,$$

and this common factor could be determined by the elementary theory of the greatest common measure of two polynomials, and would be rational in x. The original equation would therefore not be irreducible.

Of the m roots of the equation we confine our attention to one, namely y. The relations which hold between y and the other roots are of the greatest importance in the theory of functions, but we are in no way concerned with them at present.

4. Elementary functions which are not rational or algebraical are called *elementary transcendental functions*, or elementary transcendents. They include all the remaining functions which are of ordinary occurrence in elementary analysis.

The trigonometrical (or circular) and hyperbolic functions, direct and inverse, may all be expressed in terms of exponential or logarithmic functions by means of the ordinary formulae of elementary trigonometry. Thus, for example,

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}), \quad \sinh x = \frac{1}{2}(e^x - e^{-x}),$$

$$\tan^{-1} x = \frac{1}{2i}\log\left(\frac{1+ix}{1-ix}\right), \quad \tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

There was therefore no need to specify them particularly in our definition.

The elementary transcendents have been further classified in a manner first indicated by Liouville*. According to him a function is a transcendent *of the first order* if the signs of the operations of exponentiation or of the taking of logarithms which are present in the formula which defines it apply only to rational or algebraical functions. For example

$$xe^{-x^2}, \quad e^{\sqrt{x}} + e^x\sqrt{(\log x)}$$

are of the first order; and so is

$$\tan^{-1}\frac{y}{\sqrt{(1+x^2)}},$$

where y is defined by the equation

$$y^5 - y - x = 0;$$

* *Journal de Mathématiques*, t. II. (1837), p. 56.

and so is the function y defined by the equation

$$y^5 - y - e^x \log x = 0.$$

An elementary transcendent *of the second order* is one defined by a formula in which the exponentiations and takings of logarithms are applied to rational or algebraical functions or to transcendents of the first order. This class of functions includes many of great interest and importance, of which the simplest are

$$e^{e^x}, \ \log \log x.$$

It also includes the irrational or complex power of x, since *e.g.*

$$x^{\sqrt{2}} = e^{\sqrt{2} . \log x}, \ x^{1+i} = e^{(1+i) \log x};$$

the function $$x^x = e^{x \log x},$$

and the logarithms of the circular functions.

It is of course presupposed that a transcendent of the second kind is incapable of expression as one of the first kind or as a rational or algebraical function. Any rational function $R(x)$ can of course be expressed in the form

$$e^{\log R(x)}.$$

It is obvious that we can in this way proceed to define transcendents of the nth order for all values of n. Thus

$$\log \log \log x, \ \log \log \log \log x, \ \ldots\ldots$$

are of the third, fourth, ... orders.

Of course a similar classification of algebraical functions can be and has been made. Thus we may say that

$$\sqrt{x}, \ \sqrt{(x + \sqrt{x})}, \ \sqrt{\{x + \sqrt{(x + \sqrt{x})}\}}, \ \ldots\ldots$$

are algebraical functions of the first, second, third, ... orders. But the fact that there is a general theory of algebraical equations and therefore of *implicit* algebraical functions has deprived this classification of most of its importance. There is no such general theory of transcendental equations, and therefore we shall not rank as 'elementary' functions defined by transcendental equations such as

$$y = x \log y,$$

but incapable (as Liouville* has shown that in this case y is incapable) of finite explicit expression in terms of x.

* *Journal de Mathématiques*, t. III. p. 523.

5. The preceding analysis of elementary transcendental functions rests on the following theorems:

(*a*) e^x is not an algebraical function of x;

(*b*) $\log x$ is not an algebraical function of x;

(*c*) $\log x$ is not expressible in finite terms by means of signs of exponentiation and of algebraical operations, explicit or implicit (*e.g.* it is not equal to e^y, where y is any algebraical function of x);

(*d*) transcendental functions of the first, second, third, ... orders actually exist.

These theorems are quite fundamental in analysis, and are of the utmost importance for our present purpose. A proof of (*a*) and (*b*) will be given later (v. 9), but limitations of space will prevent us from giving detailed proofs of the remaining two. Liouville has given interesting extensions of some of these theorems: he has, for example, proved the impossibility of the exponential function satisfying any equation of the form

$$Ae^{\alpha p} + Be^{\beta p} + \dots + Re^{\rho p} = S,$$

where p, A, B, ..., R, S are any algebraical functions of x and α, β, ..., ρ any constants. It is not a little surprising that the necessity of giving some proof of the theorems (*a*)—(*d*) should be so generally overlooked by writers on elementary analysis.

III. The integration of elementary functions. Summary of results.

In the following pages we shall be exclusively concerned with the question of the integration of *elementary* functions. We shall endeavour to give as complete an account as the space at our disposal permits of the progress which has been made by mathematicians towards the solution of the following two problems:—

(i) *if $f(x)$ is an elementary function, how can we determine whether its integral is also an elementary function?*

(ii) *if the integral is an elementary function, how can we find it?*

Complete answers to these questions have not and probably never will be given. But sufficient has been done to give us a tolerably complete insight into the nature of the answers, and to ensure that it shall not be difficult to find the complete answers in any particular case which is at all likely to occur in elementary analysis or in its applications.

It will probably be well for us at this point to summarise the principal results which have been obtained.

1. The integral of a rational function (IV.) is *always* an elementary function. It is either itself rational or is the sum of a rational function and of a finite number of constant multiples of logarithms of rational functions (IV. 1).

If certain constants which are the roots of an algebraical equation are treated as known quantities the form of the integral can always be completely determined. But as the roots of such equations are not in general capable of explicit expression in finite terms, it is not in general possible to express the integral in an absolutely explicit form, although our knowledge of its *functional* form is complete (IV. 2).

We can always determine, by means of a finite number of elementary operations which can actually be performed, whether the integral is rational or not. If it is rational, we can determine it completely by means of such operations; if not, we can determine its rational part (IV. 3. 4).

The solution of the problem in the case of rational functions may therefore be said to be complete; for the difficulty with regard to the explicit solution of algebraical equations is one not of inadequate knowledge but of proved impossibility (IV. 5).

2. The integral of an algebraical function (V.), explicit or implicit, may or may not be elementary.

If y is an algebraical function of x the integral $\int y\,dx$ (or, more generally

$$\int R(x, y)\,dx$$

where R denotes a *rational* function) is, if an elementary function, either itself algebraical, or is the sum of an algebraical function and of a finite number of constant multiples of logarithms of algebraical functions.

All algebraical functions which occur in the integral are *rational functions of x and y* (V. 8. 11).

These theorems give a precise statement of a general principle indicated by Laplace*, '*l'intégrale d'une fonction différentielle ne peut contenir d'autres quantités radicaux que celles qui entrent dans cette fonction*,' and, we may add, cannot contain *exponentials* at all. Thus it is impossible that

$$\int \frac{dx}{\sqrt{(1+x^2)}}$$

* *Théorie Analytique des Probabilités*, p. 7.

should contain e^x or $\sqrt{(1-x)}$: if they occurred in a function whose differential coefficient is $1/\sqrt{(1+x^2)}$ it could only be *apparently*, and they could be eliminated before differentiation. Laplace's principle really rests on the fact, of which it is easy enough to convince oneself by a little reflection and the consideration of a few particular cases (though to give a rigorous proof is of course quite another matter), that *differentiation will not eliminate exponentials or algebraical irrationalities.* Nor, we may add, will it eliminate logarithms except when they occur in the simple form

$$A \log \phi(x),$$

where A is a constant, and this is why logarithms can only occur in this form in the integrals of rational or algebraical functions.

We have thus a general knowledge of the form of the integral of an algebraical function, $\int y\,dx$, when it is itself an elementary function. Whether this is so or not of course depends on the nature of the equation $f(x, y) = 0$ which defines y. If this equation, when interpreted as that of a curve in the plane (x, y), represents a *unicursal* curve, *i.e.* a curve which has the maximum number of double points possible for a curve of its degree, or whose *deficiency* is zero, x and y can be expressed simultaneously as rational functions of a third variable t, and the integral can be reduced by a substitution to that of a rational function (v. 2—5). In this case, therefore, the integral is always an elementary function. But this condition, though sufficient, is not necessary. It is *in general* true that if $f(x, y) = 0$ is not unicursal the integral is not an elementary function but a new transcendent, and we are able to classify these transcendents according to the deficiency of the curve. If, for example, the deficiency is unity, the integral is in general a new transcendent of the kind known as *elliptic integrals*, whose characteristic is that they can be transformed into integrals containing no other irrationality than the square root of a polynomial of the third or fourth degree (v. 13—15). But there are infinitely many cases in which the integral can be expressed by algebraical functions and logarithms. Similarly there are infinitely many cases in which integrals associated with curves whose deficiency is greater than unity are in reality reducible to elliptic integrals. Such abnormal cases have formed the subject of many exceedingly interesting researches, but no general method has been devised by which we can always tell, after a finite series of operations, whether any given integral is really elementary, or elliptic, or belongs to a higher order of transcendents (v. 12).

When $f(x, y) = 0$ is unicursal we can carry out the integration *completely* in exactly the same sense as in the case of rational functions. In particular, if the integral is *algebraical* it can be found by means only of elementary operations which are always practicable. And it has been shown, more generally, that we can always determine by means of such operations whether the integral of any given algebraical function is algebraical or not, and evaluate the integral when it is algebraical. And although the general problem of determining whether any given integral is an elementary function, and calculating it if it is one, has not been solved, the solution in the particular case in which the deficiency of the curve $f(x, y) = 0$ is unity is as complete as there is reason to suppose that any possible solution can be (V. 12).

3. The theory of the integration of transcendental functions (VI.) is naturally much less complete, and the number of classes of such functions for which general methods of integration exist is very small. These few classes are, however, of extreme importance in applications (VI. 2. 3).

There is a general theorem concerning the form of an integral of a transcendental function (when it is itself an elementary function) which is quite analogous to those already stated for rational and algebraical functions. The general statement of this theorem will be found in VI. (5); it shows, for instance, that the integral of a rational function of (say) x, e^x and $\log x$ is either itself a rational function of those functions, or is the sum of such a rational function and of a finite number of numerical multiples of logarithms of similar functions. From this may be deduced a number of more precise results concerning more particular forms of integrals, such as

$$\int ye^x\,dx, \quad \int y\log x\,dx,$$

where y is an algebraical function of x (VI. 4. 6).

IV. Rational functions.

1. It is proved in treatises on Algebra* that any polynomial

$$Q(x) = b_0x^n + b_1x^{n-1} + \dots + b_n$$

can be expressed in the form

$$b_0(x-a_1)^{m_1}(x-a_2)^{m_2}\dots(x-a_r)^{m_r},$$

where $m_1, \dots$ are positive integers whose sum is n, and $a_1, \dots$ are real

* See, *e.g.*, Chrystal's *Algebra*, vol. I. pp. 151—162, 248—254.

or complex quantities; and that any rational function $R(x)$, whose denominator is $Q(x)$, may be expressed in the form

$$A_0 x^p + A_1 x^{p-1} + \dots + A_p + \sum_{s=1}^{r} \left\{ \frac{\beta_{s,1}}{x - a_s} + \frac{\beta_{s,2}}{(x-a_s)^2} + \dots + \frac{\beta_{s,m_s}}{(x-a_s)^{m_s}} \right\}.$$

Hence

$$\int R(x)\,dx = A_0 \frac{x^{p+1}}{p+1} + A_1 \frac{x^p}{p} + \dots + A_p x + C$$

$$+ \sum_{s=1}^{r} \left\{ \beta_{s,1} \log (x - a_s) - \frac{\beta_{s,2}}{x - a_s} - \dots - \frac{\beta_{s,m_s}}{(m_s - 1)(x - a_s)^{m_s - 1}} \right\}.$$

From this we conclude that *the integral of any rational function is an elementary function which is rational save for the possible presence of logarithms of rational functions.* In particular the integral will be *rational* if each of the quantities $\beta_{s,1}$ is zero: this condition is evidently necessary and sufficient. A necessary but *not* sufficient condition is that $Q(x)$ should contain no simple factors.

The integral of the general rational function may be expressed in a very simple and elegant form by means of symbols of differentiation. We may suppose for simplicity that the degree of $P(x)$ is less than that of $Q(x)$; this can of course always be ensured by subtracting a polynomial from $R(x)$. Then

$$R(x) = \frac{P(x)}{Q(x)}$$

$$= \{1/(m_1-1)!\,(m_2-1)!\dots(m_r-1)!\}\, D_{m_1-1,\, m_2-1, \dots m_r-1} \frac{P(x)}{Q_0(x)},$$

where $$Q_0(x) = b_0 (x - a_1)(x - a_2)\dots(x - a_r),$$ ← should have exponents as has Q(x) on preceding page

and $D_{m_1-1,\dots,m_r-1}$ represents the operation

$$\partial^{n-r}/\partial a_1^{m_1-1}\, \partial a_2^{m_2-1} \dots \partial a_r^{m_r-1}.$$

Now $$\frac{P(x)}{Q_0(x)} = \sum_{s=1}^{r} \frac{P(a_s)}{(x - a_s)\, Q_0'(a_s)},$$

and so

$$\int R(x)\,dx$$

$$= \{1/(m_1-1)!\dots(m_r-1)!\} \frac{\partial^{n-r}}{\partial a_1^{m_1-1}\dots \partial a_r^{m_r-1}} \left[\sum_{s=1}^{r} \frac{P(a_s)}{Q_0'(a_s)} \log(x - a_s) \right].$$

For example

$$\int \frac{dx}{\{(x-a)(x-b)\}^2} = \frac{\partial^2}{\partial a \partial b} \left\{ \frac{1}{a-b} \log \left(\frac{x-a}{x-b} \right) \right\}.$$

It has been assumed above that if

$$F(x, a) = \int f(x, a)\, dx,$$

then

$$\frac{\partial F}{\partial a} = \int \frac{\partial f}{\partial a}\, dx.$$

The first equation *means* $f = \dfrac{\partial F}{\partial x}$ and the second means $\dfrac{\partial f}{\partial a} = \dfrac{\partial^2 F}{\partial x \partial a}$. As it follows from the first that $\dfrac{\partial f}{\partial a} = \dfrac{\partial^2 F}{\partial a \partial x}$ what has really been assumed is that

$$\frac{\partial^2 F}{\partial a \partial x} = \frac{\partial^2 F}{\partial x \partial a}.$$

It is known that this equation is always true for $x = x_0$, $a = a_0$ if a circle can be drawn in the plane of (x, a) whose centre is x_0, a_0 and within which the differential coefficients are continuous.

2. From one point of view the preceding investigation is complete. From others, and notably from that of practical applicability, it is far from perfect, for the simple reason that the factors of the denominator cannot be found, as the roots of $Q(x) = 0$ are not in general explicit algebraical functions of the coefficients. The difficulty may be stated thus: the *functional form* of the integral is completely determined, but it involves *constants* which cannot be expressed explicitly as functions of the constants which occur in the subject of integration. Hence we cannot determine, by the method of decomposition into partial fractions, such an integral as

$$\int \frac{4x^9 + 21x^6 + 2x^3 - 3x^2 - 3}{(x^7 - x + 1)^2}\, dx,$$

or even determine whether the integral is rational or not, although it is in reality a very simple function. A high degree of importance therefore attaches to the further problem of determining the integral of a given rational function so far as possible in an absolutely explicit form and by means of operations which are always practicable.

It is easy to see that a complete solution of this problem cannot be looked for.

Suppose for example that $P(x)$ reduces to unity, and that $Q(x) = 0$ is an equation of the fifth degree whose roots $a_1, a_2, \ldots a_5$ are all distinct, and not capable of explicit algebraical expression.

Then

$$\int R(x)\, dx = \sum_1^5 \frac{\log(x - a_s)}{Q'(a_s)}$$

$$= \log \prod_1^5 \{(x - a_s)^{1/Q'(a_s)}\},$$

and it is only if at least two of the quantities $Q'(a_s)$ are commensurable that any two or more of the factors $(x-a_s)^{1/Q'(a_s)}$ can be associated so as to give a single term of the type $A \log S(x)$, where $S(x)$ is rational. In general this will not be the case, and so it will not be possible to express the integral in any finite form which does not explicitly involve the roots. A more precise result in this connection will be proved later (IV. 5).

3. The first and most important part of the problem has been solved by Hermite, who has shown that the *rational part* of the integral can always be determined without a knowledge of the roots of $Q(x)$, and indeed without the performance of any operations other than those of elementary algebra*.

Hermite's method depends upon a fundamental theorem in elementary algebra which is also of immense importance in the ordinary theory of partial fractions†:

'If X_1 and X_2 are two polynomials in x which have no common factor, and X_3 any third polynomial, we can determine two polynomials A_1, A_2 such that

$$A_1X_1 + A_2X_2 \equiv X_3.\text{'}$$

Suppose that $$Q(x) = Q_1Q_2^2Q_3^3 \dots Q_t^t,$$

$Q_1, \dots$ denoting polynomials which have only simple roots and of which no two have any common factor. We can always determine $Q_1, \dots$ by elementary methods, as is shown in the elements of the Theory of Equations‡.

We can determine B and A_1 so that

$$BQ_1 + A_1Q_2^2Q_3^3 \dots Q_t^t \equiv P,$$

and therefore so that

$$R(x) = \frac{P}{Q} = \frac{A_1}{Q_1} + \frac{B}{Q_2^2Q_3^3 \dots Q_t^t}.$$

By a repetition of this process we can express $R(x)$ in the form

$$\frac{A_1}{Q_1} + \frac{A_2}{Q_2^2} + \dots + \frac{A_t}{Q_t^t},$$

and the problem of the integration of $R(x)$ is reduced to that of the integration of a function

$$\frac{A}{Q^\nu},$$

* The following account of Hermite's method is in substance taken from Goursat's *Cours d'Analyse Mathématique*, t. I. pp. 238—241.

† See Chrystal's *Algebra*, vol. I. pp. 119 *et seq.*

‡ Burnside and Panton, *Theory of Equations*, pp. 158—9.

where Q is a polynomial whose roots are all distinct. Since this is so, Q and its derived function Q' have no common factor; we can therefore determine C and D so that

$$CQ + DQ' \equiv A.$$

Therefore

$$\begin{aligned}\int \frac{A}{Q^\nu}\,dx &= \int \frac{CQ+DQ'}{Q^\nu}\,dx \\ &= \int \frac{C}{Q^{\nu-1}}\,dx - \frac{1}{\nu-1}\int D\,\frac{d}{dx}\left(\frac{1}{Q^{\nu-1}}\right)dx \\ &= -\frac{D}{(\nu-1)\,Q^{\nu-1}} + \int \frac{E}{Q^{\nu-1}}\,dx,\end{aligned}$$

where $E = C + \frac{1}{\nu-1}\,D'$. Proceeding in this way, and reducing by unity at each step the power of $1/Q$ which figures under the sign of integration, we ultimately arrive at an equation

$$\int \frac{A}{Q^\nu}\,dx = R_\nu(x) + \int \frac{S}{Q}\,dx,$$

where R_ν is a rational function and S a polynomial. The integral on the right-hand side has *no* rational part, since all the roots of Q are simple*. Thus the rational part of $\int R(x)\,dx$ is

$$R_2(x) + R_3(x) + \ldots + R_t(x),$$

and it has been determined without the need of any calculations other than those involved in the addition, multiplication and division of polynomials. The operation of forming the derived function of a given polynomial can of course be effected by a combination of these operations.

4. (i) Let us consider, for example, the integral

$$\int \frac{4x^9+21x^6+2x^3-3x^2-3}{(x^7-x+1)^2}\,dx,$$

mentioned above. We require polynomials A_1, A_2 such that

$$A_1(x^7-x+1) + A_2(7x^6-1) = 4x^9+21x^6+2x^3-3x^2-3.$$

These polynomials may be found in a systematic manner by means of the process for determining the greatest common divisor of x^7-x+1 and $7x^6-1$†; but the process is laborious and inconvenient. It is therefore better to use the method of undetermined coefficients. In general, if X_1 is of degree m_1 and X_2 of degree m_2, and X_3 of degree less than m_1+m_2, we can suppose A_1

* We assume for the moment that no sum of the type $\Sigma A_k \log(x-a_k)$, where all the a's are different, can be wholly or partly rational. See v. 9 (ii).

† Chrystal's *Algebra* (*loc. cit.*).

and A_2 to be of degrees m_2-1 and m_1-1 respectively, as we have then exactly m_1+m_2 equations to determine m_1+m_2 unknown coefficients. These equations are independent. For if not we could find two distinct formulae

$$A_1X_1+A_2X_2=X_3, \quad B_1X_1+B_2X_2=X_3,$$

and so
$$(A_1-B_1)X_1+(A_2-B_2)X_2=0;$$

which is impossible, since X_1 and X_2 have no common factor. The coefficients can therefore be uniquely determined. If X_3 is of degree higher than m_1+m_2-1 we must first divide it by X_1X_2 and then express the remainder in the required form.

In this case we may suppose A_1 of degree 5 and A_2 of degree 6, and we find that

$$A_1=-3x^2, \quad A_2=x^3+3.$$

Thus the rational part of the integral is

$$-\frac{x^3+3}{x^7-x+1},$$

and since $-3x^2+(x^3+3)'\equiv 0$ there is no transcendental part.

(ii) The following problem is instructive: *to find the conditions that*

$$\int\frac{ax^2+2\beta x+\gamma}{(Ax^2+2Bx+C)^2}\,dx$$

may be rational, and to determine the integral when it is rational.

We can determine p, q and r so that

$$p(Ax^2+2Bx+C)+2(qx+r)(Ax+B)\equiv ax^2+2\beta x+\gamma,$$

and the integral becomes

$$p\int\frac{dx}{Ax^2+2Bx+C}-\int(qx+r)\frac{d}{dx}\left(\frac{1}{Ax^2+2Bx+C}\right)dx$$
$$=-\frac{qx+r}{Ax^2+2Bx+C}+(p+q)\int\frac{dx}{Ax^2+2Bx+C};$$

the condition that the integral should be rational is therefore $p+q=0$ Equating coefficients we find

$$A(p+2q)=a, \quad B(p+q)+Ar=\beta, \quad Cp+2Br=\gamma.$$

Hence we deduce

$$p=-\frac{a}{A}, \qquad q=\frac{a}{A}, \qquad r=\frac{\beta}{A},$$

and $A\gamma+Ca=2B\beta$. The condition required is therefore that the two quadratics (a, β, γ), (A, B, C) should be harmonically related, and in this case

$$\int\frac{ax^2+2\beta x+\gamma}{(Ax^2+2Bx+C)^2}\,dx=-\frac{ax+\beta}{A(Ax^2+2Bx+C)}.$$

If we replace B by $(A\gamma + C\alpha)/2\beta$, and operate on both sides of the last equation with the operator

$$-\frac{1}{2}\left(\alpha'\frac{\partial}{\partial A}+\gamma'\frac{\partial}{\partial C}\right),$$

where α' and γ' are arbitrary, we deduce that

$$\int\frac{(\alpha x^2+2\beta x+\gamma)(\alpha_1 x^2+2\beta_1 x+\gamma_1)}{(Ax^2+2Bx+C)^3}\,dx$$

is rational if $\alpha_1=\alpha'\beta$, $2\beta_1=\alpha'\gamma+\gamma'\alpha$, $\gamma_1=\gamma'\beta$, or (what is the same thing) if $\alpha x^2+2\beta x+\gamma$ and $\alpha_1 x^2+2\beta_1 x+\gamma_1$ are harmonically related. By a repetition of this argument we can prove that

$$\int\frac{(\alpha x^2+2\beta x+\gamma)(\alpha_1 x^2+2\beta_1 x+\gamma_1)\dots(\alpha_n x^2+2\beta_n x+\gamma_n)}{(Ax^2+2Bx+C)^{n+2}}\,dx$$

is rational if all the quadratics are harmonically related to any one of those in the numerator.

5. It appears from the preceding paragraphs that we can always find the rational part of the integral, and can find the complete integral if the roots of $Q(x)=0$ can be found. The question is naturally suggested as to the maximum of information which can be obtained about the logarithmic part of the integral in the general case in which the factors of the denominator cannot be determined explicitly. For there are polynomials which, although they cannot be completely resolved into such factors, can nevertheless be partially resolved. For example

$$x^{14}-2x^8-2x^7-x^4-2x^3+2x+1=(x^7+x^2-1)(x^7-x^2-2x-1),$$

$$x^{14}-2x^8-2x^7-2x^4-4x^3-x^2+2x+1$$
$$=\{x^7+x^2\sqrt{2}+x(\sqrt{2}-1)-1\}\{x^7-x^2\sqrt{2}-x(\sqrt{2}+1)-1\}.$$

The factors of the first polynomial have rational coefficients: in the language of the theory of equations, the polynomial is *reducible in the rational domain*. The second polynomial is reducible in the domain formed by the *adjunction* of the single irrational $\sqrt{2}$ to the rational domain*.

We may suppose that every possible decomposition of $Q(x)$ of this nature has been made, so that

$$Q=Q_1 Q_2 \cdots Q_t.$$

Then we can resolve $R(x)$ into a sum of partial fractions of the type

$$\int\frac{P_\nu}{Q_\nu}\,dx,$$

* See Cajori, *An introduction to the Modern Theory of Equations* (Macmillan, 1904).

and so we need only consider integrals of the type

$$\int \frac{P}{Q} dx,$$

where no further resolution of Q is possible (in technical language Q is *irreducible by the adjunction of any algebraical irrationality*).

Suppose that this integral can be evaluated in a form involving only constants which can be explicitly expressed in terms of the constants which occur in P/Q. It must be of the form

$$A_1 \log X_1 + \ldots + A_k \log X_k,$$

where the A's are constants and the X's polynomials. We can suppose that no X has a multiple root: if *e.g.* X_1 had one we could determine it rationally in terms of the coefficients of X_1 and the corresponding factor $(x-\alpha)^m$ could be removed from X_1 by inserting a new term

$$mA_1 \log (x-\alpha)$$

in the expression of the integral*. For a similar reason we can suppose that no two X's have any common factor.

Now

$$\frac{P}{Q} = A_1 \frac{X_1'}{X_1} + A_2 \frac{X_2'}{X_2} + \ldots + A_k \frac{X_k'}{X_k},$$

or
$$P X_1 X_2 \ldots X_k = Q \Sigma A_\nu X_1 \ldots X_{\nu-1} X_\nu' X_{\nu+1} \ldots X_k.$$

All the terms under the sign of summation are divisible by X_1 save the first, which is prime to X_1. Hence Q must be divisible by X_1: and similarly, of course, by $X_2, X_3, \ldots, X_k$. Since P is prime to Q, $X_1 X_2 \ldots X_k$ is divisible by Q: hence

$$Q = X_1 X_2 \ldots X_k$$

save for a constant factor. But *ex hypothesi* Q is not resoluble into factors which contain only explicit algebraical irrationalities. Hence all the X's save one must reduce to constants, and so P must be a constant multiple of Q', and

$$\int \frac{P}{Q} dx = A \log Q,$$

where A is a constant. Unless this is the case the integral cannot be expressed in a form involving only constants explicitly expressed in terms of the constants which occur in P and Q.

* If X_1 had *more than one* multiple root *of the same order* we might not be able actually to determine them rationally in terms of its coefficients (*e.g.* $X_1 = (x-\alpha)(x^5 - x - a)^2$), but we could so determine the factor corresponding to all these roots, so that the argument would not be affected.

Thus, for instance, the integral

$$\int \frac{dx}{x^5+ax+b}$$

cannot be expressed in a form involving only constants explicitly expressed in terms of a and b; and

$$\int \frac{5x^4+c}{x^5+ax+b}\,dx$$

can be so expressed if and only if $c=a$. We thus confirm an inference formed before (IV. 2) in a less rigid way.

Before quitting this part of our subject we may consider one further problem: under what circumstances is

$$\int R(x)\,dx = A \log R_1(x)$$

where A is a constant and R_1 rational? Since the integral has no rational part it is clear that $Q(x)$ must have only simple factors, and that the degree of $P(x)$ must be less than that of $Q(x)$. We may therefore use the formula

$$\int R(x)\,dx = \log \prod_1^r \{(x-a_s)^{P(a_s)/Q'(a_s)}\}.$$

The necessary and sufficient condition is that all the quantities $P(a_s)/Q'(a_s)$ must be commensurable. If *e.g.*

$$R(x) = \frac{x-\gamma}{(x-a)(x-\beta)},$$

$(a-\gamma)/(a-\beta)$ and $(\beta-\gamma)/(\beta-a)$ must be commensurable, *i.e.* $(a-\gamma)/(\beta-\gamma)$ must be a rational number. If the denominator is given we can find all the values of γ which are admissible: for $\gamma=(aq-\beta p)/(q-p)$ where p and q are integers.

V. Algebraical Functions.

1. We shall now consider the integrals of algebraical functions, explicit or implicit. The theory of the integration of such functions is far more extensive and difficult than was the case with the rational functions, and we can only give here a brief account of the most important results and of the most obvious of their applications.

If $y_1, y_2, \dots, y_n$ are algebraical functions of x any algebraical function z of $x, y_1, \dots, y_n$ is an algebraical function of x. This is obvious if we confine ourselves to *explicit* algebraical functions. In the general case we have a number of equations of the type

$$P_{\nu,0}(x)\,y_\nu^{m_\nu} + P_{\nu,1}(x)\,y_\nu^{m_\nu-1} + \dots + P_{\nu,m_\nu}(x) = 0$$

$(\nu=1, 2, \dots, n)$, and

$$P_0(x, y_1, \dots y_n)\,z^m + \dots + P_m(x, y_1, \dots, y_n) = 0,$$

where the P's represent polynomials in their arguments. The elimination of $y_1, y_2, \dots, y_n$ between these equations gives an equation for z, whose coefficients are polynomials in x only.

The importance of this from our present point of view lies in the fact that we may consider the standard algebraical integral under any of the forms

(a) $\int y\,dx$, where $f(x, y)=0$;

(b) $\int R(x, y)\,dx$, where $f(x, y)=0$ and R is rational;

(c) $\int R(x, y_1, \dots, y_n)\,dx$, where $f_1(x, y)=0, \dots, f_n(x, y_n)=0$.

It is, for example, much more convenient to treat such an irrational as

$$\frac{x-\sqrt{(x+1)}-\sqrt{(x-1)}}{1+\sqrt{(x+1)}+\sqrt{(x-1)}}$$

as a rational function of x, y_1, y_2, where $y_1=\sqrt{(x+1)}$, $y_2=\sqrt{(x-1)}$, $y_1^2=x+1$, $y_2^2=x-1$, than as a rational function of x and

$$y=\sqrt{(x+1)}+\sqrt{(x-1)},$$

so that

$$y^4-4xy^2+4=0;$$

while to treat it as a simple irrational y, so that our fundamental equation is

$$(x-y)^4-4x(x-y)^2(1+y)^2+4(1+y)^4=0$$

is evidently still more inconvenient.

Before we proceed to consider the general form of the integral of an algebraical function it will be convenient to consider one most important case in which the integral can be immediately reduced to that of a rational function, and therefore is always an elementary function itself. It will perhaps be well at this point to emphasize two points which we have already mentioned (II. 3): viz. (i) that our defining relation $f(x, y)=0$ is always supposed to be *irreducible* and (ii) that we confine our attention to *one* of its roots.

2. The class of integrals alluded to immediately above is that covered by the following theorem.

If there is a variable t connected with x and y (or $y_1, y_2, \dots, y_n$) *by rational relations*

$$x=R_1(t), \quad y=R_2(t)$$

(or $y_1=R_2^{(1)}(t)$, $y_2=R_2^{(2)}(t)$, ...) *the integral*

$$\int R(x, y)\,dx$$

(or $\int R(x, y_1, \dots, y_n)\,dx$) *can be evaluated in finite terms by means of elementary functions.*

This is practically obvious, since

$$R(x, y) = R\{R_1(t), R_2(t)\} = S(t),$$

$$\frac{dx}{dt} = R_1'(t) = T(t),$$

all the capital letters denoting rational functions.

It is to be observed that from our present point of view it is quite immaterial whether an integral be transformed by a real or by an imaginary substitution. For the equation

$$\int f(x)\, dx = \int f\{\phi(t)\}\, \phi'(t)\, dt,$$

means simply that if

$$\frac{dF(x)}{dx} = f(x),$$

then

$$\frac{dF\{\phi(t)\}}{dt} = f\{\phi(t)\}\, \phi'(t),$$

and it is of no importance whether ϕ is a *real* function or not.

It is of the utmost importance, of course, when we are dealing with *definite* integrals.

The most important case of this theorem is that in which x and y are connected by the general quadratic relation

$$(a, b, c, f, g, h \between x, y, 1)^2 = 0.$$

The integral can be made rational in an infinite number of ways. For suppose that (ξ, η) is any point on the conic, and that

$$(y - \eta) = t(x - \xi)$$

is any line through the point. If we eliminate y between these equations we obtain an equation of the second degree in x,

$$T_0 x^2 + 2T_1 x + T_2 = 0,$$

T_0, T_1, T_2 being polynomials in t. But one root of this equation must be ξ, which is independent of t; and when we divide by $x - \xi$ we obtain an equation of the *first* degree for the abscissa of the variable point of intersection, in which the coefficients are again polynomials in t. Hence this abscissa is a rational function of t; the ordinate of the variable point of intersection is also a rational function of t, and as t varies this point coincides with every point of the conic in turn. In fact the equation of the conic may be written in the form

$$au^2 + 2huv + bv^2 + 2(a\xi + h\eta + g)u + 2(h\xi + b\eta + f)v = 0,$$

where $u = x - \xi$, $v = y - \eta$, and the other point of intersection of the line $v = tu$ and the conic is given by

$$x = \xi - \frac{2\{a\xi + h\eta + g + t(h\xi + b\eta + f)\}}{a + 2ht + bt^2},$$

$$y = \eta - \frac{2t\{a\xi + h\eta + g + t(h\xi + b\eta + f)\}}{a + 2ht + bt^2}.$$

The most important case is that in which $b = -1, f = h = 0$, so that

$$y^2 = ax^2 + 2gx + c.$$

The integral is then made rational by the substitution

$$x = \xi - \frac{2(a\xi + g - t\eta)}{a - t^2}, \quad y = \eta - \frac{2t(a\xi + g - t\eta)}{a - t^2},$$

where ξ, η are any quantities such that

$$\eta^2 = a\xi^2 + 2g\xi + c.$$

We may for instance suppose $\xi = 0$, $\eta = \sqrt{c}$; or $\eta = 0$, while ξ is a root of the equation $a\xi^2 + 2g\xi + c = 0$.

3. It must not be imagined that this general method is always *practically* the best for the integration of

$$\int R(x, \sqrt{ax^2 + 2gx + c})\, dx.$$

In practice we proceed as follows. Let

$$y = \sqrt{X} = \sqrt{ax^2 + 2gx + c}.$$

Then $R(x, y)$ is of the form $P(x, y)/Q(x, y)$, where P and Q are polynomials. By means of the equation $y^2 = ax^2 + 2gx + c$, $R(x, y)$ may be reduced to the form

$$\frac{A + B\sqrt{X}}{C + D\sqrt{X}} = \frac{(A + B\sqrt{X})(C - D\sqrt{X})}{C^2 - D^2 X},$$

where A, B, C, D are polynomials in x; and so to the form $M + N\sqrt{X}$, where M and N are rational, or (what is the same thing) the form

$$P + \frac{Q}{\sqrt{X}},$$

where P and Q are rational. In the cases of most frequent occurrence in practice a, g, c, $\sqrt{ax^2 + 2gx + c}$ and the coefficients which occur in P and Q are *real*. The rational part may be integrated by the methods of IV., and the integral $\int \frac{Q}{\sqrt{X}}\, dx$ may by the theory of partial fractions be made to depend upon a number of integrals of functions of the forms

$$\frac{1}{(x-p)\sqrt{X}}, \quad \frac{1}{(x-p)^r\sqrt{X}},$$

$$\frac{\xi x + \eta}{(ax^2 + 2\beta x + \gamma)\sqrt{X}}, \quad \frac{\xi x + \eta}{(ax^2 + 2\beta x + \gamma)^r\sqrt{X}};$$

where p, ξ, η, α, β, γ are real constants and r a positive integer. The result is generally required in an explicitly real form: and as further progress depends on transformations involving p (or α, β, γ) it is generally not advisable to break up a quadratic factor $\alpha x^2+2\beta x+\gamma$ whose roots are imaginary into its constituent linear factors.

The integrals which involve powers of $x-p$ or $\alpha x^2+2\beta x+\gamma$ higher than the first may be deduced from those which involve only the first powers by differentiations with respect to p or γ.

The integral
$$\int \frac{dx}{(x-p)\sqrt{X}},$$
may be evaluated in a variety of manners.

(i) We may follow the general method described above, taking
$$\xi=p, \quad \eta=\sqrt{(ap^2+2gp+c)}*.$$

Eliminating y from the equations
$$y^2=ax^2+2gx+c, \quad y-\eta=t(x-\xi),$$
and dividing by $x-\xi$, we obtain
$$t^2(x-\xi)+2\eta t-a(x+\xi)-2g=0,$$
and so
$$-\frac{2dt}{t^2-a}=\frac{dx}{t(x-\xi)+\eta}=\frac{dx}{y}.$$

Hence
$$\int \frac{dx}{(x-\xi)y}=-2\int \frac{dt}{(x-\xi)(t^2-a)}.$$

But
$$(t^2-a)(x-\xi)=2a\xi+2g-2\eta t;$$
and so
$$\int \frac{dx}{(x-p)y}=-\int \frac{dt}{a\xi+g-\eta t}=\frac{1}{\eta}\log(a\xi+g-\eta t)$$
$$=\frac{1}{\sqrt{(ap^2+2gp+c)}}\log\{t\sqrt{(ap^2+2gp+c)}-ap-g\}.$$

If $ap^2+2gp+c<0$ the transformation is imaginary†.

Suppose, *e.g.*, (*a*) $y=\sqrt{(x+1)}$, $p=0$, (*b*) $y=\sqrt{(x-1)}$, $p=0$. We find

(*a*)
$$\int \frac{dx}{x\sqrt{(x+1)}}=\log(t-\tfrac{1}{2}),$$
where
$$t^2x+2t-1=0,$$
or
$$t=(-1+\sqrt{x+1})/x,$$

* Jordan, *Cours d'Analyse*, t. II. p. 21.

† We have supposed that p is not a root of the equation $X=0$. If it is, the integral is, as we shall see later (v. 9 (i)), algebraical, and can be determined by a series of elementary algebraical operations which are always practicable. Otherwise the integral is purely transcendental. A factor of the denominator of Q which is also a factor of X can be found by elementary methods, and the algebraical part of $\int \frac{Q}{\sqrt{X}}dx$ can always be determined completely by such methods. This result is quite analogous to that already proved in the case of rational functions.

the positive sign of the radical corresponding to the case in which

$$y = +\sqrt{(x+1)}:$$

(b) $$\int \frac{dx}{x\sqrt{(x-1)}} = \frac{1}{i}\log(it - \tfrac{1}{2}),$$

where $$t^2 x + 2it - 1 = 0.$$

Neither of these results is expressed in the most convenient form, the second in particular being very inconvenient.

(ii) The most straightforward method of procedure is to use the substitution

$$x - p = \frac{1}{z},$$

commonly used in text-books on the Integral Calculus. We then obtain

$$\int \frac{dx}{(x-p)y} = \int \frac{dz}{\sqrt{a_1 z^2 + 2g_1 z + c_1}},$$

which is a well known form reducible by a substitution of the type $z = t + k$ to one of the three standard forms

$$\int \frac{dt}{\sqrt{(m^2 - t^2)}}, \quad \int \frac{dt}{\sqrt{(t^2 + m^2)}}, \quad \int \frac{dt}{\sqrt{(t^2 - m^2)}}.$$

These forms may of course be rationalised, as *e.g.* by the respective substitutions

$$t = \frac{2mu}{1+u^2}, \quad t = \frac{2mu}{1-u^2}, \quad t = \frac{m(1+u^2)}{2u}.$$

but it is more convenient to use the transcendental substitutions

$$t = m\sin\phi, \quad t = m\sinh\phi, \quad t = m\cosh\phi.$$

And it is often convenient when dealing with more complicated algebraical integrals, containing only one irrationality of one of the types

$$\sqrt{(m^2 - t^2)}, \quad \sqrt{(t^2 + m^2)}, \quad \sqrt{(t^2 - m^2)},$$

to reduce it to a transcendental form not involving roots by means of one of these three substitutions. As alternative substitutions

$$t = m\tanh\phi, \quad t = m\tan\phi, \quad t = m\sec\phi,$$

are often useful. Prof. Bromwich points out that the forms usually given in the text-books for these three standard integrals, viz.

$$\sin^{-1}(x/a), \quad \sinh^{-1}(x/a), \quad \cosh^{-1}(x/a),$$

are not entirely accurate. It is obvious, for example, that the first two of these functions are odd functions of a, while the corresponding integrals are even functions of a. The correct formulae are $\sin^{-1}(x/|a|)$, $\sinh^{-1}(x/|a|)$, and $$\pm\cosh^{-1}(|x|/|a|) = \log(x + \sqrt{x^2 - a^2}),$$

where the ambiguous sign is the same as that of x, as the reader will easily verify. In some ways it is more convenient to use the equivalent forms

$$\tan^{-1}\left(\frac{x}{\sqrt{a^2 - x^2}}\right), \quad \tanh^{-1}\left(\frac{x}{\sqrt{a^2 + x^2}}\right), \quad \tanh^{-1}\left(\frac{\sqrt{x^2 - a^2}}{x}\right).$$

(iii) The most elegant method of integration is unquestionably that associated with the name of Prof. Greenhill*, who uses the transformation

$$y=\frac{\sqrt{(ax^2+2gx+c)}}{x-p}.$$

It will be found that

$$\int\frac{dx}{(x-p)\sqrt{X}}=\int\frac{dz}{\sqrt{\{(ap^2+2gp+c)z^2+g^2-ac\}}},$$

which is one of the three standard forms written above.

When we are dealing with the integral

$$\int\frac{\xi x+\eta}{(ax^2+2\beta x+\gamma)\sqrt{X}}\,dx,$$

(which will naturally only be the case when the roots of $ax^2+2\beta x+\gamma=0$ are imaginary) by far the most convenient method of procedure is to use Prof. Greenhill's substitution

$$z=\sqrt{\frac{X}{(ax^2+2\beta x+\gamma)}}=\sqrt{\frac{X}{X_1}},$$

say. If $$J\equiv(a\beta-ga)x^2-(ca-a\gamma)x+g\gamma-c\beta,$$

$$\frac{1}{z}\frac{dz}{dx}=\frac{J}{XX_1}\qquad\text{.....(1)}.$$

The maximum and minimum values of z are given by $J=0$.

Again $$z^2-\lambda=\frac{(a-\lambda a)x^2+2(g-\lambda\beta)x+c-\lambda\gamma}{X_1};$$

wherein the numerator will be a perfect square if

$$K\equiv(a\gamma-\beta^2)\lambda^2-(a\gamma+ca-2g\beta)\lambda+ac-g^2=0.$$

It will be found by a little calculation that the discriminant of this quadratic and that of $J=0$ differ from one another and from

$$(\xi_1-\xi_1')(\xi_2-\xi_1')(\xi_1-\xi_2')(\xi_2-\xi_2'),$$

where ξ_1, ξ_2 are the roots of $X=0$ and ξ_1', ξ_2' those of $X_1=0$, only by a constant factor which is always negative. Since ξ_1' and ξ_2' are conjugate imaginaries this product is positive, and so $J=0$ and $K=0$ have real roots. We denote the roots of the latter by

$$\lambda_1,\ \lambda_2\ (\lambda_1>\lambda_2).$$

Then $$\lambda_1-z^2=\frac{\{x\sqrt{(\lambda_1 a-a)}+\sqrt{(\lambda_1\gamma-c)}\}^2}{X_1}=\frac{(mx+n)^2}{X_1}\qquad\text{.....(2)}$$

$$z^2-\lambda_2=\frac{\{x\sqrt{(a-\lambda_2 a)}+\sqrt{(c-\lambda_2\gamma)}\}^2}{X_1}=\frac{(m'x+n')^2}{X_1}\qquad\text{.....(2')}$$

say. Further, since $z^2-\lambda$ can vanish for two equal values of x only if λ is equal to λ_1 or λ_2, *i.e.* when z is a maximum or a minimum, J can only differ from

$$(mx+n)(m'x+n')$$

* A. G. Greenhill, *A Chapter in the Integral Calculus* (1888, Francis Hodgson), p. 12; *Differential and Integral Calculus*, p. 399.

by a constant factor; and by comparing coefficients and using the identity

$$(\lambda_1 a - a)(a - \lambda_2 a) = (a\beta - ga)^2/(a\gamma - \beta^2),$$

we find that $$J = \sqrt{(a\gamma - \beta^2)}\,(mx+n)\,(m'x+n') \qquad (3).$$

Finally we can write $\xi x + \eta$ in the form

$$A\,(mx+n) + B\,(m'x+n').$$

Using equations (1), (2), (2′), (3) we find that

$$\int \frac{\xi x + \eta}{X_1 \sqrt{X}}\, dx = \int \frac{A\,(mx+n) + B\,(m'x+n')}{J} \sqrt{X_1}\, dz$$
$$= \frac{A}{\sqrt{(a\gamma - \beta^2)}} \int \frac{dz}{\sqrt{(\lambda_1 - z^2)}} + \frac{B}{\sqrt{(a\gamma - \beta^2)}} \int \frac{dz}{\sqrt{(z^2 - \lambda_2)}},$$

and the integral is expressed in terms of real standard forms*.

4. We may now proceed to consider the general case to which the theorem of IV. 2 applies. It will be convenient to recall two well known definitions in the theory of algebraical plane curves. A curve of degree n can have at most $\frac{1}{2}(n-1)(n-2)$ double points†. If the actual number of double points is ν the number

$$p = \tfrac{1}{2}(n-1)(n-2) - \nu$$

is called the *deficiency*‡ of the curve.

If the coordinates x, y of the points on a curve can be expressed *rationally* in terms of a parameter t by equations

$$x = R_1(t), \quad y = R_2(t),$$

we shall say that the curve is *unicursal*. In this case we have seen that we can always evaluate

$$\int R\,(x, y)\, dx$$

in finite terms.

The fundamental theorem in this part of our subject is

'*A curve whose deficiency is zero is unicursal, and vice versa.*'

Suppose first that the curve possesses the maximum number of double points§. Since

$$\tfrac{1}{2}(n-1)(n-2) + n - 3 = \tfrac{1}{2}(n-2)(n+1) - 1,$$

* The reader should refer to Prof. Greenhill's writings quoted above and to Chrystal's *Algebra*, vol. I. pp. 464 *et seq.* Prof. Greenhill gives interesting numerical examples.

† Salmon, *Higher Plane Curves*, p. 29.

‡ Salmon, *ibid.* p. 29. French *genre*, German *Geschlecht*.

§ We suppose in what follows that the singularities of the curve are all ordinary double points. The necessary modifications when this is not the case are not difficult to make. It has been shown that an ordinary multiple point of order k may be regarded as equivalent to $\frac{1}{2}k(k-1)$ ordinary double points (Salmon, *loc. cit.* p. 28, and $\frac{1}{2}(n-2)(n+1)$ points are just sufficient to determine a curve of degree $n-2$* we can, through the $\frac{1}{2}(n-1)(n-2)$ double points and $n-3$ other points chosen arbitrarily on the curve draw a *simply infinite* set of curves of degree $n-2$, which we may suppose to have the equation

$$g(x, y) + t\,h(x, y) = 0,$$

where t is a variable parameter. Any one of these curves meets the given curve in $n(n-2)$ points of which $(n-1)(n-2)$ are accounted for by the $\frac{1}{2}(n-1)(n-2)$ double points and $n-3$ by the $n-3$ arbitrarily chosen points. These

$$(n-1)(n-2) + n - 3 = n(n-2) - 1$$

points are independent of t; and so there is but *one* point of intersection which depends on t. The coordinates of this point are given by

$$g(x, y) + t\,h(x, y) = 0, \quad f(x, y) = 0.$$

The elimination of y gives an equation of degree $n(n-2)$ in x whose coefficients are polynomials in t, and but one root of this equation varies with t. The eliminant is therefore divisible by a factor of degree $n(n-2) - 1$ which does not contain t. There remains a simple equation in x whose coefficients are polyn[illegible] Thus the x-coordinate of the variable point is [illegible]mined as a rational function of t, and the y-coordinate [illegible]similarly determined.

We may therefor[illegible]

$$x = R_1(t), \quad y = R_2(t).$$

[illegible]e reduce these fractions to the same denominator we can [illegible]xpress the coordinates in the form

$$x = \frac{\phi_1(t)}{\phi_3(t)}, \quad y = \frac{\phi_2(t)}{\phi_3(t)} \quad \text{..................} (1),$$

where ϕ_1, ϕ_2, ϕ_3 are polynomials which have no common factor. The polynomials will in general be of degree n; none of them can be of

Basset, *Quarterly Journal*, XXXVI. p. 360). A curve of degree n which has an ordinary multiple point of order $n-1$ (equivalent to $\frac{1}{2}(n-1)(n-2)$ ordinary double points) is therefore unicursal.

The theory of higher plane curves abounds in puzzling particular cases which have to be fitted into the general theory by more or less obvious conventions, and to give a satisfactory account of a complicated compound singularity is sometimes by no means easy. The investigation which follows must be regarded as essentially occupied with the *general* case.

* Salmon, *loc. cit.* p. 16.

higher degree, and one at least must be actually of that degree, since an arbitrary straight line

$$\lambda x + \mu y + \nu = 0$$

must cut the curve in exactly n points*.

We shall now prove the second part of the theorem. If

$$x : y : 1 :: \phi_1(t) : \phi_2(t) : \phi_3(t),$$

where ϕ_1, ϕ_2, ϕ_3 are polynomials of degree n, the line

$$ux + vy + w = 0$$

will meet the curve in n points whose parameters are given by

$$u\phi_1(t) + v\phi_2(t) + \phi_3(t) = 0.$$

This equation will have a double root t_0 if

$$u\phi_1(t_0) + v\phi_2(t_0) + \phi_3(t_0) = 0,$$

$$u\phi_1'(t_0) + v\phi_2'(t_0) + \phi_3'(t_0) = 0.$$

Hence the equation of the tangent at the point t_0 is

$$\begin{vmatrix} x & y & 1 \\ \phi_1(t_0) & \phi_2(t_0) & \phi_3(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \phi_3'(t_0) \end{vmatrix} = 0.$$

If (x, y) is a fixed point this may be regarded as an equation to determine the parameters of the points of contact of the tangents from (x, y). Now

$$\phi_2(t_0)\,\phi_3'(t_0) - \phi_2'(t_0)\,\phi_3(t_0)$$

is of degree $2n-2$ in t_0, the coefficient of t_0^{2n-1} obviously vanishing. Hence in general the number of tangents which can be drawn to a unicursal curve from a fixed point (the *class* of the curve) is $2n-2$. But the class of a curve whose only singular points are δ double points is known† to be $n(n-1)-2\delta$. Hence the number of double points is

$$\tfrac{1}{2}\{n(n-1)-(2n-2)\} = \tfrac{1}{2}(n-1)(n-2).$$

5. The preceding argument fails if $n<3$, but we have already seen that all conics are unicursal. The case next in importance is that of

* See Niewenglowski's *Géométrie Analytique*, t. II. p. 103. By way of illustration of the remark concerning particular cases in the footnote (§) to page 25, the reader will do well to consider the example given by Niewenglowski in which

$$x = \frac{t^2}{t^2-1}, \quad y = \frac{t^2+1}{t^2-1};$$

equations which appear to represent the straight line $2x = y + 1$ (part of the line only, if we consider only real values of t).

† Salmon, *Higher Plane Curves*, p. 54.

a cubic with a double point. If the double point is not at infinity we can, by a change of origin, reduce the equation of the curve to the form

$$(ax+by)(cx+dy)=px^3+3qx^2y+3rxy^2+sy^3,$$

and by considering the intersections of the curve with the line $y=tx$ we find

$$x=\frac{(a+bt)(c+dt)}{p+3qt+3rt^2+ps}, \quad y=\frac{t(a+bt)(c+dt)}{p+3qt+3rt^2+ps}.$$

If the double point is at infinity the equation of the curve is of the form

$$(\alpha x+\beta y)^2(\gamma x+\delta y)+\epsilon x+\zeta y+\theta=0$$

(the curve having a pair of parallel asymptotes), and by considering the intersection of the curve with the line $\alpha x+\beta y=t$ we find

$$x=-\frac{\delta t^3+\zeta t+\beta\theta}{(\beta\gamma-\alpha\delta)t^2+\epsilon\beta-\alpha\zeta}, \quad y=\frac{\gamma t^3+\epsilon t+\alpha\theta}{(\beta\gamma-\alpha\delta)t^3+\epsilon\beta-\alpha\zeta}.$$

(i) The case next in complexity is that of a quartic with three double points.

(*a*) The lemniscate $\quad (x^2+y^2)^2=a^2(x^2-y^2)$

has three double points, the origin and the circular points at infinity. The circle

$$x^2+y^2=t(x-y)$$

passes through these points and one other fixed point at the origin, as it touches the curve there. Solving we find

$$x=\frac{a^2t(t^2+a^2)}{t^4+a^4}, \quad y=\frac{a^2t(t^2-a^2)}{t^4+a^4}.$$

(*b*) The curve $\quad 2ay^3-3a^2y^2=x^4-2a^2x^2$

has the double points $(0, 0)$, (a, a), $(-a, a)$. Using the auxiliary conic

$$x^2-ay=tx(y-a)$$

we find $\quad x=\frac{a}{t^3}(2-3t^2), \quad y=\frac{a}{2t^4}(2-3t^2)(2-t^2).$

(ii) The curve $\quad y^n=x^n+ax^{n-1}$

has a multiple point of order $n-1$ at the origin, and is therefore unicursal. In this case it is sufficient to consider the intersection of the curve with the line $y=tx$. This may be harmonised with the general theory by regarding the curve

$$y^{n-3}(y-tx)=0,$$

as passing through each of the $\frac{1}{2}(n-1)(n-2)$ double points collected at the origin and through $n-3$ other fixed points collected at the point

$$y=0, \quad x=-a.$$

The curves

$$y^n = x^n + ax^{n-1} \qquad\qquad (1),$$
$$y^n = 1 + az \qquad\qquad (2),$$

are projectively equivalent, as appears by rendering their equations homogeneous by the introduction of quantities $z=1$ in (1) and $x=1$ in (2). We conclude that (2) is unicursal, having the maximum number of double points at infinity. In fact we may put

$$y = t, \quad az = t^n - 1;$$

and

$$\int R\{z, \sqrt[n]{(1+az)}\}\, dz$$

is integrable in finite terms.

(*c*) The curve

$$y^m = A(x-a)^\mu (x-b)^\nu,$$

is unicursal if and only if either (i) μ or $\nu = 0$ or (ii) $\mu + \nu = m$. Hence

$$\int R\{x, \sqrt[m]{(x-a)^\mu (x-b)^\nu}\}\, dx,$$

is integrable in finite terms for all forms of R in these two cases only; of course it is integrable for special forms of R in other cases*.

6. There is of course a similar theory connected with *unicursal curves in space of any number of dimensions.* Consider for example the integral

$$\int R\{x, \sqrt{(ax+b)}, \sqrt{(cx+d)}\}\, dx.$$

A linear substitution $x = lx + m$ reduces this to the form

$$\int R_1\{y, \sqrt{(y+2)}, \sqrt{(y-2)}\}\, dy,$$

and this can be rationalised by taking

$$y = t^2 + \frac{1}{t^2}, \quad \sqrt{(y+2)} = t + \frac{1}{t}, \quad \sqrt{(y-2)} = t - \frac{1}{t}.$$

The curve whose Cartesian coordinates ξ, η, ζ are given by

$$\xi : \eta : \zeta : 1 :: t^4 + 1 : t(t^2+1) : t(t^2-1) : t^2,$$

is a unicursal twisted quartic, the intersection of the parabolic cylinders

$$\xi = \eta^2 - 2, \quad \xi = \zeta^2 + 2.$$

It is easy to deduce that

$$\int R\left\{x, \sqrt{\left(\frac{ax+b}{mx+n}\right)}, \sqrt{\left(\frac{cx+d}{mx+n}\right)}\right\} dx$$

can always be evaluated in finite form.

7. When the deficiency of the curve $f(x, y) = 0$ is not zero the integral

$$\int R(x, y)\, dx$$

is *in general* not an elementary function; and the consideration of such integrals has consequently introduced a whole series of classes of

* Ptaszycki, *Bull. des Sciences Mathématiques*, XII. p. 263: Appell and Goursat, *Théorie des Fonctions Algébriques*, p. 245.

new transcendents into analysis. The simplest case is that in which the deficiency is *unity*: in this case, as we shall see later on, the integrals are expressible in terms of elementary functions and certain new transcendents known as elliptic integrals. When the deficiency rises above unity the integration necessitates the introduction of new transcendents of growing complexity.

But there are infinitely many particular cases in which integrals, associated with curves whose deficiency is unity or greater than unity, can be expressed in terms of elementary functions, or are even algebraical themselves. For instance the deficiency of

$$y^2 = 1 + x^3$$

is unity. But

$$\int \frac{x+1}{x-2} \frac{dx}{\sqrt{(1+x^3)}} = 3 \log \frac{(1+x)^2 - 3\sqrt{(1+x^3)}}{(1+x)^2 + 3\sqrt{(1+x^3)}},$$

$$\int \frac{2-x^3}{1+x^3} \frac{dx}{\sqrt{(1+x^3)}} = \frac{2x}{\sqrt{(1+x^3)}}.$$

And, before we say anything concerning the new transcendents to which integrals of this class in general give rise, we shall consider what has been done in the way of formulating rules to enable us to identify such cases and to assign the form of the integral when it can be expressed in finite terms. It will be as well to say at once that this problem has not been completely solved.

8. The first general theorem deals with the case in which the integral is algebraical, and asserts that *if*

$$u = \int y\,dx$$

is an algebraical function of x it is a rational function of x and y.

If u is an algebraical function of x it satisfies an equation

$$\psi(x, u) = 0,$$

whose coefficients are polynomials in x. By means of the equation $f(x, y) = 0$ we can introduce y into this equation and write it in the form

$$\phi(x, y, u) = 0,$$

without altering the degree of the equation in u.

The succeeding proof depends essentially on the presence of y explicitly in this equation. If

$$f(x, y) \equiv P_0(x) y^n + \dots + P_n(x) = 0,$$

and Ax^k is a term in $P_n(x)$, it is obvious that

$$Ax^k = P(x, y),$$

P denoting a polynomial. If $\psi(x, u)$ contains a power of x as high as the kth we can obviously introduce y at once by means of this equation: if not we must first multiply $\psi(x, u)$ by some power of x.

We can suppose $\phi(x, y, u)$ irreducible, for if not we could replace it by some simpler equation.

By differentiating $f = 0$, $\phi = 0$ we obtain

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0, \qquad \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} + \frac{\partial \phi}{\partial u}\frac{du}{dx} = 0,$$

and eliminating $\frac{dy}{dx}$ we obtain an expression for $\frac{du}{dx}$ of the form

$$\frac{du}{dx} = \frac{\lambda(x, y, u)}{\mu(x, y, u)},$$

where λ and μ are polynomials. In order that u shall be the integral of y it is necessary and sufficient that $\frac{du}{dx} = y$, *i.e.* that the equations

$$\phi(x, y, u) = 0,$$

$$\lambda(x, y, u) - y\mu(x, y, u) = 0,$$

shall hold simultaneously.

Now the equation $\phi = 0$ has other roots $u_1, u_2, \ldots, u_k$ besides u (unless it is of the first degree, in which case u is obviously a rational function of x and y), and these roots must all satisfy the two equations $\phi = 0$, $\lambda - y\mu = 0$. For otherwise we could determine the greatest common measure of ϕ and $\lambda - y\mu$, considered as polynomials in u: this common factor would be a polynomial in x, y, u and divide $\phi(x, y, u)$. But this is impossible, since $\phi(x, y, u)$ is irreducible.

Hence $u, u_1, u_2, \ldots u_k$ are *all* integrals of y, and therefore

$$\frac{1}{k+1}(u + u_1 + \ldots + u_k)$$

is an integral of y. But this function is a *symmetric* function of the roots of $\phi(x, y, u) = 0$, and is therefore a rational function of x and y. The theorem is therefore proved.

Thus

$$\int y\,dx = \frac{P(x, y)}{Q(x, y)},$$

if the integral is algebraical, P and Q being polynomials. If $y_1, y_2, \ldots, y_{n-1}$ are the roots of $f(x, y) = 0$, other than y,

$$\int y\,dx = \frac{P(x, y)\,Q(x, y_1) \ldots\ldots Q(x, y_{n-1})}{Q(x, y)\,Q(x, y_1) \ldots\ldots Q(x, y_{n-1})}.$$

The denominator is a symmetric function of $y, y_1, \ldots, y_{n-1}$ and therefore a rational function of x. Moreover

$$Q(x, y_1)\, Q(x, y_2) \ldots Q(x, y_{n-1})$$

is a symmetric function of the roots of the equation in z

$$\frac{f(x, z)}{z - y} = 0,$$

whose coefficients are polynomials in x and y of which the first does not contain y. It is therefore a rational function of x and y integral with respect to y, and so $\int y\,dx$ consists of a sum of a number of terms of the type $R_\nu(x)\, y^\nu$. By means of the equation $f(x, y) = 0$ all such terms which involve powers of y higher than the nth can be eliminated. We thus arrive at the final conclusion that *if $\int y\,dx$ is algebraical it may be expressed in the form*

$$R_0 + R_1 y + \ldots + R_{n-1} y^{n-1}$$

*where R_0, R_1, ... are rational functions of x**.

The most important case is that in which

$$y = \sqrt[n]{R(x)},$$

where $R(x)$ is rational. In this case

$$y^n = R(x),$$

$$\frac{dy}{dx} = \frac{R'(x)}{ny^{n-1}}.$$

But

$$y = R_0' + R_1' y + \ldots + R'_{n-1} y^{n-1}$$
$$+ \{R_1 + 2R_2 y + \ldots + (n-1) R_{n-1} y^{n-2}\} \frac{dy}{dx} \quad \ldots\ldots(1).$$

Eliminating $\frac{dy}{dx}$ between these equations we obtain an equation

$$\varpi(x, y) = 0$$

where $\varpi(x, y)$ is a polynomial. In virtue of a theorem proved and used before this equation, and therefore the equation (1), must be satisfied by *all* the roots of $y^n = R(x)$. The same therefore holds of the equation

$$\int y\,dx = R_0 + R_1 y + \ldots + R_{n-1} y^{n-1}.$$

In this equation we may therefore replace y by ωy, ω being any

* For the preceding proof see Abel, *Œuvres*, t. I. p. 545 *et seq.*, and *Crelle*, *b.* IV. p. 264; Liouville, *Journal de l'École Polytechnique*, t. XIV. p. 149; Bertrand, *Calcul Integral*, Ch. V.

primitive nth root of unity. Making this substitution and multiplying by ω^{n-1} we obtain

$$\int y\,dx = \omega^{n-1}R_0 + R_1 y + \omega R_2 y + \ldots + \omega^{n-2}R_{n-1}y^{n-1},$$

and on adding the n equations of this type we obtain

$$\int y\,dx = R_1 y.$$

Thus in this case the functions R_0, R_2, ..., R_{n-1} all disappear.

It has been shown by Liouville that the preceding results enable us in all cases to obtain by a finite number of elementary algebraical operations a solution of the problem '*to determine whether* $\int y\,dx$ *is algebraical, and to find the integral when it is algebraical.*'

9. (i) It would take too long to attempt to trace in detail the steps of the general argument. We shall confine ourselves to a solution of a particular problem which will give an illustration sufficient for our present purpose of the general nature of the arguments which must be employed.

We shall determine under what circumstances

$$\int \frac{dx}{(x-p)\sqrt{(ax^2+2gx+c)}},$$

is algebraical. This question might of course be answered by actually evaluating the integral in the general case and finding when the integral function reduces to an algebraical function. We are now, however, in a position to answer it without any such integration.

In this case $\quad y = \dfrac{1}{\sqrt{X}}, \quad X = (x-p)^2(ax^2+2gx+c),$

and if $\int y\,dx$ is algebraical it must be of the form $R(x)/\sqrt{X}$. Hence

$$y = \frac{d}{dx}\left(\frac{R}{\sqrt{X}}\right),$$

or

$$2X = 2XR' - RX'.$$

We can now show that R is a polynomial in x. For if $R = U/V$, where U and V are polynomials, V, if not a mere constant, must contain a factor

$$(x+A)^a, \quad (a>0),$$

and we can put

$$R = \frac{U}{W(x+A)^a},$$

where U and W do not contain the factor $x+A$. Substituting this expression for R, and reducing, we obtain

$$\frac{2aUWX}{x+A} = 2U'WX - 2UW'X - UWX' - 2W^2X(x+A)^a.$$

Hence X must be divisible by $x+A$.

Suppose then that $\quad X = (x+A)^p X$

where $\bar{X}$ is prime to $x+A$. Substituting in the equation last obtained we deduce

$$\frac{(2a+p)\,UW\bar{X}}{x+A}=2U'W\bar{X}-2UW'\bar{X}-UW\bar{X}'-2W^2\bar{X}(x+A)^{a},$$

which is obviously impossible, since neither U, W, nor $\bar{X}$ is divisible by $x+A$.

Hence $$\int\frac{dx}{(x-p)\sqrt{(ax^2+2gx+c)}}=\frac{U(x)}{(x-p)\sqrt{(ax^2+2gx+c)}}$$

where $U(x)$ is a polynomial. Differentiating and clearing of radicals

$$\{(x-p)(U'-1)-U\}(ax^2+2gx+c)=U(x-p)(ax+g).$$

If the first term in U is Ax^m we find at once on equating coefficients of x^{m+2} that $m=2$. We may therefore take

$$U=Ax^2+2Bx+C,$$

so that

$$\{(x-p)(2Ax+2B-1)-Ax^2-2Bx-C\}(ax^2+2gx+c)$$
$$=(x-p)(ax+g)(Ax^2+2Bx+C).$$

Now $ax^2+2gx+c$ is not divisible by $ax+g$, as in that case it would be a perfect square. Hence either $ax^2+2gx+c$ and $Ax^2+2Bx+C$ differ only by a constant factor, or they have one factor $x-q$ in common, and $x-p$ divides $ax^2+2gx+c$. If $x-t$ is the second factor of $Ax^2+2Bx+C$ we must have

$$2Ax+2B-1=2A(x-t).$$

Dividing out by $A(x-p)(x-q)(x-t)$ we obtain

$$a\{2(x-p)-(x-q)\}=ax+g,$$

or $a(q-2p)=g$, *i.e.* $2q-4p=-q-p$, $q=p$, which is not the case. Hence the only possible case is that in which

$$\int\frac{dx}{(x-p)\sqrt{(ax^2+2gx+c)}}=\text{const}\,\frac{\sqrt{(ax^2+2gx+c)}}{x-p},$$

where $ap^2+2gp+c=0$. It is easily verified that this equation is actually satisfied, the value of the constant being $1/\sqrt{(g^2-ac)}$. The formula is equivalent to

$$\int\frac{dx}{(x-p)\sqrt{(x-p)(x-q)}}=\frac{2}{q-p}\sqrt{\left(\frac{x-q}{x-p}\right)}\,^{*}.$$

(ii) The result of the preceding paragraph also enables us to supply a strict proof of the two fundamental theorems stated without proof in II. 5; viz.

(*a*) e^x is not an algebraical function of x:

(*b*) $\log x$ is not an algebraical function of x.

* Greenhill, *A Chapter in the Integral Calculus*, p. 18. The same method may be applied to the integral $\displaystyle\int\frac{dx}{(x-p)^r\sqrt{(ax^2+2gx+c)}}\ (r>1)$.

If $y=\log x$, $x=e^y$, and if x is an algebraical function of y, y is an algebraical function of x. It is therefore sufficient to prove that

$$y=\int\frac{dx}{x}$$

is not algebraical. If y is algebraical it is a rational function of x and $1/x$, *i.e.* of x. That is to say

$$\log x = X_1/X_2 \quad\ldots\ldots(1)$$

where X_1 and X_2 are polynomials. It is not difficult to show by purely algebraical reasoning that the equation

$$\frac{1}{x}=\frac{X_1'X_2-X_1X_2'}{X_2^2},$$

obtained by differentiating (1), is impossible. But it is simpler to argue otherwise. The right-hand side of (1) either tends to a finite limit for $x=\infty$ or becomes infinite or vanishes like a power of x, viz. x^{m-n}, where m and n are the degrees of X_1 and X_2. On the other hand $\log x$ tends to infinity with x, but more slowly than any power of x. Hence $\log x$ is not rational, and therefore not algebraical.

Not only is it impossible that $\log x$ should be algebraical but also it is impossible that any sum of the form $\Sigma A_k \log(x-a_k)$, where all the a's are different, should be algebraical (and therefore, by v. 8, rational). The reader should by now be able to prove this for himself, or he can refer to Liouville's proof of this and a number of more general theorems in the memoir referred to on p. 5. It is this result which was assumed in IV. 3.

(iii) If $$y^3-3y+2x=0$$

the integral $\int y\,dx$ is algebraical and equal to

$$\tfrac{1}{8}(6xy-3y^2)^*.$$

10. The general theorem of 8 gives the first step in the rigid proof of *Laplace's principle* stated in III. 2. On account of the immense importance of this principle we repeat Laplace's words—'*l'integrale d'une fonction différentielle ne peut contenir d'autres quantités radicaux que celles qui entrent dans cette fonction.*' This general principle, combined with arguments similar to those used above (v. 9 (i)) in a particular case, enables us to prove without difficulty that a great many integrals cannot be algebraical, notably the standard elliptic integrals

$$\int\frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},\qquad \int\sqrt{\left(\frac{1-x^2}{1-k^2x^2}\right)}\,dx,\qquad \int\frac{dx}{\sqrt{(4x^3-g_2x-g_3)}},$$

which give rise by inversion to the *elliptic functions.*

* This is easy to verify. A synthetic proof following Liouville's general line of argument will be found in a memoir by Raffy (*Annales de l'École Normale Supérieure*, p. 185).

11. We must now consider in a very summary manner the much more difficult question of the nature of those integrals of algebraical functions which are expressible in finite terms by means of the elementary transcendental functions. In the first place, *no integral of any algebraical function can contain any exponential.* Of this theorem it is, as we remarked before, easy to become convinced by a little reflection, as doubtless did Laplace, who certainly possessed no rigid proof. The reader will find little difficulty in coming to the conclusion that exponentials cannot be eliminated from an elementary function by differentiation. But we would strongly recommend him to study the exceedingly beautiful and ingenious proof of this proposition given by Liouville*. We have unfortunately no space to insert it here.

It is instructive to consider particular cases of this theorem. Suppose for example that $\int y dx$, where y is algebraical, were a polynomial in x and e^x, say

$$\Sigma\Sigma a_{m,n} x^m e^{nx} \qquad (1).$$

When this expression is differentiated e^x must disappear from it: otherwise we should have an algebraical relation between x and e^x. Expressing the conditions that the coefficient of every power of e^x in the differential coefficient of (1) vanishes identically we find that the same must be true of (1), so that after all the integral does not really contain e^x. Liouville's proof is in reality a development of this idea.

The integral of an algebraical function (if expressible in finite terms) can therefore only contain algebraical or logarithmic functions. The next step is to show that the logarithms can only be logarithms of the first order, *i.e.* simple logarithms of algebraical functions, and can only enter linearly, so that the general integral must be of the type

$$\int y dx = t + A \log u + B \log v + \ldots + K \log w,$$

where A, B,..., K are constants and t, u, v,.., w algebraical functions. Only when the logarithms occur in this simple form will differentiation eliminate them†.

Lastly it can be shown† by arguments similar to those of 8 that t, u,..., w are rational functions of x and y. Thus $\int y dx$, if a finite elementary function, is *the sum of a rational function of x and y and of certain constant multiples of logarithms of such functions.* We can suppose that no two of A, B,... K are commensurable, or indeed, more generally, that no linear relation

$$A\alpha + B\beta + \ldots + K\kappa = 0,$$

* *Journal de l'École Polytechnique*, t. XIV. cahier XXIV. p. 46. The proof may also be found in Bertrand's *Calcul Intégral*, p. 99.

† Liouville and Bertrand (*loc. cit.*).

with rational coefficients, holds between them. For if such a relation held we could eliminate A from the integral, writing it in the form

$$\int y\,dx = t + B\log\left(vu^{-\frac{\beta}{\alpha}}\right) + \ldots + K\log\left(wu^{-\frac{\kappa}{\alpha}}\right).$$

The case of greatest interest is that in which y is a rational function of x and $\sqrt{X}$, where X is a polynomial. As we have already seen, y can in this case be expressed in the form

$$P + \frac{Q}{\sqrt{X}},$$

where P and Q are rational functions of x. We shall suppress the rational part and suppose that $y = Q/\sqrt{X}$. In this case the general theorem gives

$$\int \frac{Q}{\sqrt{X}}\,dx = S + \frac{T}{\sqrt{X}} + A\log(\alpha + \beta\sqrt{X}) + B\log(\gamma + \delta\sqrt{X}) + \ldots,$$

where S, T, α, β, γ, δ, ... are rational. If we differentiate this equation we obtain an algebraical identity in which we can change the sign of $\sqrt{X}$. Thus we may change the sign of $\sqrt{X}$ in the integral equation. If we do this and subtract we obtain (after writing $2A, \ldots$ for $A, \ldots$)

$$\int \frac{Q}{\sqrt{X}}\,dx = \frac{T}{\sqrt{X}} + A\log\frac{\alpha + \beta\sqrt{X}}{\alpha - \beta\sqrt{X}} + B\log\frac{\gamma + \delta\sqrt{X}}{\gamma - \delta\sqrt{X}} + \ldots,$$

which is the standard form for such an integral. It is evident that we may suppose α, β, γ, ... to be *polynomials*.

12. (i) By means of this theorem it is possible to prove that a number of important integrals, notably the integrals

$$\int \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}, \qquad \int \sqrt{\left\{\frac{1-x^2}{1-k^2x^2}\right\}}\,dx, \qquad \int \frac{dx}{\sqrt{(4x^3 - g_2x - g_3)}},$$

are not explicitly expressible in finite terms, and so represent genuinely new transcendents. The formal proof of this was worked out by Liouville*; it rests merely on a consideration of the possible forms of the differential coefficients of expressions of the form

$$\frac{T}{\sqrt{X}} + A\log\frac{\alpha + \beta\sqrt{X}}{\alpha - \beta\sqrt{X}} + \ldots,$$

and the arguments used are purely algebraical and of no great theoretical difficulty. The proof is however too detailed to be inserted here. It is not difficult to find shorter proofs, but these are of a less elementary character, being based on ideas drawn from the theory of functions†.

* *Journal de l'École Polytechnique*, t. XIV. cahier XXIII. p. 37.

† The proof given by Laurent (*Traité d'Analyse*, t. IV. p. 153) appears at first sight to combine the advantages of both methods of proof but unfortunately will not stand a closer examination.

The general questions of this nature which arise in connection with integrals of the form $\int \frac{Q}{\sqrt{X}} dx$ (or, more generally, $\int \frac{Q}{\sqrt[m]{X}} dx$) are of extreme interest and difficulty. The case which has received most attention is that in which $m=2$ and X is of the third or fourth degree, in which case the integral is said to be *elliptic*. An integral of this kind is called *pseudo-elliptic* if it is expressible in terms of algebraical and logarithmic functions. An example was given above (v. 7). General methods have been given for the construction of such integrals, and it has been shown that certain interesting forms are pseudo-elliptic. In Goursat's *Cours d'Analyse**, for instance, it is shown that if $f(x)$ is a rational function such that

$$f(x)+f\left(\frac{1}{k^2x}\right)=0,$$

then
$$\int \frac{f(x)\,dx}{\sqrt{\{x(1-x)(1-k^2x)\}}}$$

is pseudo-elliptic. But so far no absolutely complete method has been devised by which we can always determine in a finite number of steps whether a *given* elliptic integral is pseudo-elliptic, and integrate it if it is, and there is reason to suppose that no such method *can* be given.

And up to the present it has not, so far as we know, been actually and explicitly proved that the function

$$u=\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

is not a root of an elementary transcendental equation; all that has been shown is that it is not *explicitly* expressible in terms of elementary transcendents.

The processes of reasoning employed here, and in the memoirs to which we have referred, therefore do not suffice to prove that the inverse function $x=\operatorname{sn} u$ is not an elementary function of u. Such a proof must rest on the known properties of the function $\operatorname{sn} u$, and would lie altogether outside the province of this pamphlet.

The reader who desires to pursue the subject further will find references to the original authorities in the Appendix.

(ii) One particular class of integrals which is of especial interest is that of the *binomial integrals*

$$\int x^m (ax^n+b)^p\,dx,$$

where m, n, p are *rational*. Putting $ax^n=bt$ and neglecting a constant factor we obtain an integral of the form

$$\int t^q (1+t)^p\,dt$$

where p and q are rational. If p is an integer and q a fraction r/s this can be at once integrated by putting $t=u^s$, which rationalises the integrand. If q is an integer and $p=r/s$ we put $1+t=u^s$. If $p+q$ is an integer, and $p=r/s$, we put $1+t=tu^s$.

* Pp. 264–266.

It follows from Tchebichef's researches (to which references are given in the Appendix) that these three cases are the only ones in which the integral can be evaluated in finite form.

13. In v. 4. 5 we considered in some detail the integrals connected with curves whose deficiency is zero. We shall now consider in a more summary way the case next in simplicity, that in which the deficiency is unity, so that the number of double points is

$$\tfrac{1}{2}(n-1)(n-2)-1=\tfrac{1}{2}n(n-3).$$

It has been shown by Clebsch* that in this case the coordinates of the points of the curve can be expressed as *rational functions of a parameter t and of the square root of a polynomial in t of the third or fourth degree.*

The fact is that the curves

$$y^2=a+bx+cx^2+dx^3,$$
$$y^2=a+bx+cx^2+dx^3+ex^4,$$

are the simplest curves of deficiency 1. The first is simply the typical cubic without a double point. The second is a quartic with two double points, in this case coinciding in a *tacnode* at infinity, as we see by making the equation homogeneous with z, then writing 1 for y and comparing the resulting equation with the form treated by Salmon on p. 215 of his *Higher Plane Curves.* The reader who is familiar with the theory of algebraical plane curves will remember that the deficiency of a curve is unaltered by any birational transformation of coordinates, and that any curve of deficiency 1 can be birationally transformed into the cubic whose equation is written above.

The argument by which this general theorem is proved is very much like that by which we proved the corresponding theorem for unicursal curves. The simplest case is that of the general cubic curve. We take a point on the curve as origin: then the equation of the curve is of the form

$$(a,\, b,\, c,\, d\between x,\, y)^3+(e,\, f,\, g\between x,\, y)^2+(h,\, k\between x,\, y)=0.$$

Let us consider the intersections of this curve with the secant $y=tx$. Eliminating y we see that x is given by

$$(a,\, b,\, c,\, d\between 1,\, t)^3x^2+(e,\, f,\, g\between 1,\, t)^2x+(h,\, k\between 1,\, t)=0.$$

Hence the only irrationality which enters into the expression of x, and so of y, is

$$\sqrt{\{[(e,\, f,\, g\between 1,\, t)^2]^2-4(h,\, k\between 1,\, t)[(a,\, b,\, c,\, d\between 1,\, t)^3]\}}.$$

A more elegant method has been given by Clebsch†. If we write the cubic in the form

$$LMN=P,$$

* *Crelle*, b. 64, p. 210.

† Hermite, *Cours*, pp. 422–425.

where L, M, N, P are linear functions of x and y, so that L, M, N are the asymptotes, the hyperbolas $LM = t$ will meet the cubic in four fixed points at infinity, and therefore in only two points which depend on t. For these points

$$LM = t, \quad P = tN.$$

Thus if the curve is

$$x^3 + y^3 - 3axy + 1 = 0,$$

so that

$$L = \omega x + \omega^2 y + a, \quad M = \omega^2 x + \omega y + a, \quad N = x + y + a, \quad P = a^3 - 1,$$

ω being an imaginary cube root of unity, we find that the line

$$x + y + a = (a^3 - 1)/t$$

meets the curve in the points given by

$$x = \frac{b - at}{2t} \pm \frac{\sqrt{3T}}{6t}, \qquad y = \frac{b - at}{2t} \mp \frac{\sqrt{3T}}{6t},$$

where $b = a^3 - 1$ and

$$T = 4t^3 - 9a^2t^2 + 6abt - b^2.$$

In particular for the curve

$$x^3 + y^3 + 1 = 0,$$

$$x = \frac{1}{2t\sqrt{3}}(-\sqrt{3} + \sqrt{4t^3 - 1}), \qquad y = \frac{1}{2t\sqrt{3}}(-\sqrt{3} - \sqrt{4t^3 - 1}).$$

14. It will be plain from what precedes that

$$\int R\{x, \sqrt[3]{(a + bx + cx^2 + dx^3)}\}\, dx,$$

can always be reduced to an elliptic integral, the deficiency of the cubic

$$y^3 = a + bx + cx^2 + dx^3$$

being unity.

In general integrals associated with curves whose deficiency is greater than unity cannot be so reduced. But associated with every curve of (let us say) deficiency 2 there will be an infinity of integrals

$$\int R\,(x, y)\, dx$$

reducible to elliptic integrals or even to elementary functions; and there are curves of deficiency 2 for which *all* such integrals are reducible.

For example $\int R\,(x, \sqrt{x^6 + ax^4 + bx^2 + c})\, dx$

may be split up into the sum of the integral of a rational function and two integrals of the type

$$\int \frac{R_1(x^2)\, dx}{\sqrt{(x^6 + ax^4 + bx^2 + c)}}, \qquad \int \frac{xR_2(x^2)\, dx}{\sqrt{(x^6 + ax^4 + bx^2 + c)}},$$

and each of these becomes elliptic on putting $x^2 = t$. But the deficiency of

$$y^2 = x^6 + ax^4 + bx^2 + c$$

is *two*. Another example is given by

$$\int R(x, \sqrt[4]{x^4 + ax^3 + bx^2 + cx + d})\,dx\text{*}.$$

15. It would be beside our present purpose to enter into any detail as to the general theory of elliptic integrals, still less of the integrals (usually called Abelian) associated with curves of deficiency greater than unity. We have seen that if the deficiency is unity the integral can be transformed into the form

$$\int R(x, \sqrt{X})\,dx$$

where $$X = x^4 + ax^3 + bx^2 + cx + d\,\dagger.$$

It can be shown that by a transformation of the type

$$x = \frac{\alpha t + \beta}{\gamma t + \delta}$$

this can be transformed into an integral

$$\int R(t, \sqrt{T})\,dt$$

where $$T = t^4 + At^2 + B.$$

We can then, as when T is of the second degree (v. 3) decompose this into two integrals of the forms

$$\int R(t)\,dt, \qquad \int \frac{R(t)\,dt}{\sqrt{T}},$$

of which the first is elementary while the second can be decomposed‡ into the sum of an algebraical term and certain multiples of the integrals

$$\int \frac{dt}{\sqrt{T}}, \qquad \int \frac{t^2 dt}{\sqrt{T}}$$

and of a number of integrals of the type

$$\int \frac{dt}{(t - \tau)\sqrt{T}}.$$

These integrals (v. 12 (i)) cannot in general be reduced to elementary functions, and are therefore new transcendents.

* See Legendre, *Traité des Fonctions Elliptiques*, t. I. Ch. XXVI–XXVI., XXXII–XXXIII.; Bertrand, *Calcul Intégral*, p. 67, and Enneper, *Elliptische Funktionen*, Note 1, where abundant references are given.

† There is a similar theory for curves of deficiency 2, in which X is of the *sixth* degree.

‡ v. *e.g.* Goursat, *Cours d'Analyse*, t. I. pp. 255–267.

We will only add, before leaving this part of our subject, that the *algebraical* part of these integrals can be found by means of the elementary algebraical operations, as was the case with the *rational* part of the integral of a rational function, and with the algebraical part of the simple integrals considered in v. 3 and v. 9.

VI. Transcendental functions.

1. The theory of the integration of transcendental functions is naturally much less complete than that of the integration of rational or even of algebraical functions. It is obvious from the nature of the case that this must be so, as there is no general theorem concerning transcendental functions which in any way corresponds to the theorem that any combination of algebraical functions, explicit or implicit, may be regarded as a simple algebraical function, the root of an equation of a simple standard type.

One may almost say that there *is* no general theory: the theory reduces to an enumeration of the few cases in which the integral may be transformed by an appropriate substitution into an integral of a rational or algebraical function. These few cases are however of immense importance in the applications of the general theory of integration.

2. (i) The integral

$$\int F(e^{ax}, e^{bx}, \dots, e^{kx})\, dx$$

where F is an *algebraical* function, and $a, b, \dots, k$ *commensurable* numbers can always be reduced to that of an algebraical function. In particular

$$\int R(e^{ax}, e^{bx}, \dots, e^{kx})\, dx,$$

where R is rational, can always be calculated in finite terms. In the first place a substitution of the type $x = \alpha y$ will reduce it to the form

$$\int R_1(e^y)\, dy,$$

and then the substitution $e^y = z$ reduces this to the integral of a rational function.

In particular, since $\cosh x$ and $\sinh x$ are rational functions of e^x, and $\cos x$ and $\sin x$ are rational functions of e^{ix}, the integrals

$$\int R(\cosh x, \sinh x)\, dx, \quad \int R(\cos x, \sin x)\, dx$$

can always be evaluated in finite form. In the case of the latter integral the substitution indicated above is imaginary, and it is generally more convenient to use the substitution

$$\tan \tfrac{1}{2}x = t,$$

which reduces the integral to that of a rational function, since

$$\cos x = \frac{1-t^2}{1+t^2}, \qquad \sin x = \frac{2t}{1+t^2}, \qquad dx = \frac{2dt}{1+t^2}.$$

(ii) The integrals

$$\int R(\cosh x, \sinh x, \cosh 2x, \ldots\ldots \sinh mx)\, dx,$$

$$\int R(\cos x, \sin x, \cos 2x, \ldots\ldots \sin mx)\, dx,$$

are included in the two standard integrals above.

Let us consider some further developments concerning the integral

$$\int R(\cos x, \sin x)\, dx\text{*}.$$

If we make the substitution $z = e^{ix}$ the subject of integration becomes a rational function $H(z)$, which we suppose split up into

(i) a constant and certain positive and negative powers of z,

(ii) groups of terms of the type

$$\frac{A}{z-a} + \frac{A_1}{(z-a)^2} + \ldots + \frac{A_n}{(z-a)^{n+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The terms (i) when expressed in terms of x give rise to a term

$$\Sigma\, (c_k \cos kx + d_k \sin kx).$$

In the group (1) we put $z = e^{ix}$, $a = e^{ia}$, and using the equation

$$\frac{1}{z-a} = \tfrac{1}{2}e^{-ia}\left(-1 - i \cot \frac{x-a}{2}\right)$$

we obtain a polynomial of degree $n+1$ in $\cot \frac{1}{2}(x-a)$. Since

$$\cot^2 x = -1 - \frac{d \cot x}{dx}, \qquad \cot^3 x = -\cot x - \frac{1}{2}\frac{d}{dx}(\cot^2 x), \ldots$$

this polynomial may be transformed into the form

$$C + C_0 \cot \tfrac{1}{2}(x-a) + C_1 \frac{d}{dx} \cot \tfrac{1}{2}(x-a) + \ldots + C_n \frac{d^n}{dx^n} \cot \tfrac{1}{2}(x-a).$$

The function $R(\cos x, \sin x)$ is now expressed as a sum of a number of terms each of which is immediately integrable. The integral is a rational function of $\cos x$ and $\sin x$ if all the constants C_0 vanish; otherwise it includes a number of terms of the type

$$2C_0 \log \sin \tfrac{1}{2}(x-a).$$

* Hermite, *Cours*, pp. 320 *et seq.*

Let us suppose for simplicity that $H(z)$ when split up into partial fractions contains no terms of the types

$$C, \quad z^m, \quad z^{-m}, \quad 1/(z-a)^p \qquad (p>1).$$

Then

$$R(\cos x, \sin x) = C_0 \cot \tfrac{1}{2}(x-a) + D_0 \cot \tfrac{1}{2}(x-\beta) + \ldots,$$

and the constants C_0, D_0, ... are easily assigned by considering the behaviour of the function R for values of x very nearly equal to $a, \beta, \ldots$. It is often convenient to use the equation

$$\cot \tfrac{1}{2}(x-a) = \cot(x-a) + \operatorname{cosec}(x-a)$$

which enables us to decompose the function R into two parts $U(x)$ and $V(x)$ such that

$$U(x+\pi) = U(x), \quad V(x+\pi) = -V(x).$$

If R has the period π, it is easy to see that V must vanish identically; if it merely changes sign when x is increased by π, U must vanish identically. Thus we find without difficulty that if $m<n$

$$\frac{\sin mx}{\sin nx} = \frac{1}{2n}\sum_0^{2n-1} \frac{(-)^k \sin ma}{\sin(x-a)} = \frac{1}{n}\sum_0^{n-1} \frac{(-)^k \sin ma}{\sin(x-a)}$$

where $$a = \frac{k\pi}{n},$$

or $$= \frac{1}{n}\sum_0^{n-1} (-)^k \sin ma \cot(x-a)$$

according as $m+n$ is odd or even.

(iii) One of the most important integrals in applications is

$$\int \frac{dx}{a+b\cos x},$$

where a and b are real, which may be integrated as explained above, or by the transformation $\tan \frac{1}{2}x = t$. A more elegant method is the following. If $|a|>|b|$ we suppose a positive and use the transformation

$$(a+b\cos x)(a-b\cos y) = a^2-b^2,$$

which leads to $$\frac{dx}{a+b\cos x} = \frac{dy}{\sqrt{(a^2-b^2)}}.$$

If $|a|<|b|$ we suppose b positive and use the transformation

$$(b\cos x + a)(b\cosh y - a) = b^2-a^2.$$

The integral $$\int \frac{dx}{a+b\cos x + c\sin x}$$

may be reduced to this form by the substitution $x+a=y$, where $\cot a = b/c$. The integrals

$$\int \frac{dx}{(a+b\cos x)^n}, \qquad \int \frac{dx}{(a+b\cos x+c\sin x)^n}$$

may be at once deduced by differentiation. The integral

$$\int \frac{dx}{(A\cos^2 x + 2B\cos x\sin x + C\sin^2 x)^n}$$

is really of the same type, since

$$A\cos^2 x + 2B\cos x\sin x + C\sin^2 x = \tfrac{1}{2}(A+C) + \tfrac{1}{2}(A-C)\cos 2x + B\sin 2x.$$

And similar methods may be applied to the corresponding integrals which contain hyperbolic functions, so that this type includes a large variety of integrals of common occurrence.

(iv) The same substitutions may of course be used when the subject of integration is an irrational function of $\cos x$ and $\sin x$, though sometimes it is better simply to use the substitutions $\cos x = t$ or $\sin x = t$. Thus

$$\int R(\cos x, \sin x, \sqrt{X})\,dx,$$

where $$X = (a, b, c, f, g, h \between \cos x, \sin x, 1)^2$$

is reduced to an elliptic integral by the substitution $\tan\frac{1}{2}x = t$. The most important integrals of this type are

$$\int \frac{R(\cos x, \sin x)\,dx}{\sqrt{(1-k^2\sin^2 x)}}, \qquad \int \frac{R(\cos x, \sin x)\,dx}{\sqrt{(a+b\cos x + c\sin x)}}.$$

3. The integral

$$\int P(x, e^{ax}, e^{bx}, \ldots, e^{kx})\,dx,$$

where $a, b, \ldots, k$ are *any* numbers, and P is a polynomial, can always be integrated in finite terms. For it is obvious that it can be reduced to the sum of a finite number of integrals of the type

$$\int x^p e^{Ax}\,dx;$$

and $$\int x^p e^{Ax}\,dx = \left(\frac{\partial}{\partial A}\right)^p \int e^{Ax}\,dx = \left(\frac{\partial}{\partial A}\right)^p \frac{e^{Ax}}{A}.$$

This type includes a large variety of integrals such as

$$\int x^m \cos^\mu px \sin^\nu qx\,dx, \quad \int x^m \cosh^\mu px \sinh^\nu qx\,dx,$$
$$\int x^m e^{-ax}\cos^\mu px\,dx, \quad \int x^m e^{-ax}\sin^\nu qx\,dx,$$

....

(m, μ, ν, being positive integers) for which 'formulae of reduction' are usually given in the text-books on the Integral Calculus.

Such integrals as

$$\int P(x, \log x)\,dx, \quad \int P(x, \sin^{-1}x)\,dx, \ldots,$$

where P is a polynomial, may be reduced to particular cases of the above general integral by the obvious substitutions

$$x = e^y, \quad x = \sin y, \ldots.$$

4. Except for the two classes of functions considered in the two preceding paragraphs, there are no really general classes of transcendental functions which we can *always* integrate in finite terms, although of course there are innumerable particular forms which may be integrated by particular devices. There are however many classes of such integrals for which a systematic reduction theory may be given, analogous to the reduction theory of elliptic integrals. Such a reduction theory endeavours in each case

(i) to split up any integral of the class under consideration into the sum of a number of parts of which some can be integrated in finite terms, while the others cannot;

(ii) to reduce the number of the latter terms to the minimum possible.

(iii) to prove that the terms of the latter class are incapable of further reduction, and so constitute genuinely new and independent transcendents.

As an example of this process we shall consider the integral

$$\int e^x R(x)\,dx$$

where $R(x)$ is a rational function of x*. By means of the ordinary theory of partial fractions this may be decomposed into the sum of a number of terms of the type

$$A\int\frac{e^x}{x-a}\,dx,\qquad A_m\int\frac{e^x}{(x-a)^{m+1}}\,dx,\ \ldots,\qquad B\int\frac{e^x}{x-b}\,dx,\ \ldots.$$

Since

$$\int\frac{e^x}{(x-a)^{m+1}}\,dx=-\frac{e^x}{m\,(x-a)^m}+\frac{1}{m}\int\frac{e^x}{(x-a)^m}\,dx,$$

the integral may be further reduced so as to contain only

(i) a term $$e^x S(x)$$

where $S(x)$ is a rational function;

(ii) a number of terms of the type

$$\alpha\int\frac{e^x\,dx}{x-a}.$$

If all the constants α vanish the integral can be calculated in the finite form $e^x S(x)$. If they do not we can at any rate assert that the integral cannot be calculated *in this form*. For no such relation as

$$\alpha\int\frac{e^x\,dx}{x-a}+\beta\int\frac{e^x\,dx}{x-b}+\ldots+\kappa\int\frac{e^x\,dx}{x-k}=e^x\,T(x),$$

where T is rational (or even algebraical) is possible. To see this it is

* v. Hermite, *Cours d'Analyse*, p. 352 *et seq.*

only necessary to put $x = a + h$ and to expand in ascending powers of h. For

$$a \int \frac{e^x dx}{x - a} = ae^a \int \frac{e^h}{h} dh$$
$$= ae^a (\log h + h + \ldots),$$

and no *logarithm* can occur in any of the other terms.

Consider, for example, the integral

$$\int e^x \left(1 - \frac{1}{x}\right)^3 dx.$$

This is equal to

$$e^x - 3 \int \frac{e^x}{x} dx + 3 \int \frac{e^x}{x^2} dx - \int \frac{e^x}{x^3} dx,$$

and since
$$3 \int \frac{e^x}{x^2} dx = -\frac{3e^x}{x} + 3 \int \frac{e^x}{x} dx,$$

and
$$-\int \frac{e^x}{x^3} dx = \frac{e^x}{2x^2} - \frac{1}{2} \int \frac{e^x}{x^2} dx = \frac{e^x}{2x^2} + \frac{e^x}{2x} - \frac{1}{2} \int \frac{e^x}{x} dx,$$

we obtain finally

$$\int e^x \left(1 - \frac{1}{x}\right)^3 dx = e^x \left(1 - \frac{7}{2x} + \frac{1}{2x^2}\right) - \frac{1}{2} \int \frac{e^x}{x} dx.$$

Similarly it will be found that

$$\int e^x \left(1 - \frac{2}{x}\right)^2 dx = 2e^x \left(\frac{1}{2} - \frac{2}{x}\right),$$

this integral being expressible in finite terms.

Since
$$\int \frac{e^x}{x - a} dx = e^a \int \frac{e^y}{y} dy,$$

if $x = y + a$, all integrals of this kind may be made to depend on known functions and on the single transcendent

$$\int \frac{e^x}{x} dx,$$

which is usually denoted by $\text{li}\,(e^x)$ and is of great importance in the theory of numbers. The question of course arises as to whether this integral is capable of finite expression in terms of elementary functions.

Now Liouville* has proved the following theorem: *if y is any algebraical function of x, and*

$$\int e^x y\, dx$$

is integrable in finite terms, its value will be of the form

$$e^x (\alpha + \beta y + \ldots + \lambda y^{n-1})$$

* *Crelle*, XIII. p. 107 *et seq.* Liouville shows how the integral, when of this form, may always be calculated.

$\alpha, \beta, \ldots, \lambda$ being rational functions of x and n the degree of the algebraical equation which determines y as a function of x.

Liouville's proof rests on the same general principles as do those of the corresponding theorems concerning the integral $\int y dx$. It will be observed that no logarithmic terms can occur, and that the theorem is therefore very similar to that which holds for $\int y dx$ in the simple case in which the integral is *algebraical*. The argument which shows that no logarithmic terms occur is substantially the same as that which showed that if they occur in the integral of an algebraical function they must occur *linearly*. In this case the occurrence of the exponential factor precludes even this possibility, since differentiation will not eliminate logarithms when they occur in the form

$$e^x \log f(x).$$

In particular, if y is a rational function, the integral must be of the type

$$e^x R(x)$$

and this we have already seen to be impossible. Hence the 'logarithm-integral'

$$\operatorname{li} e^x = \int \frac{e^x}{x} dx = \int^{e^x} \frac{dy}{\log y}$$

is really a new transcendent, which cannot be expressed in finite terms by means of elementary functions; and the same is true of all integrals of the type

$$\int e^x R(x) dx$$

which cannot be calculated in finite terms by means of the process of reduction sketched above.

The integrals

$$\int \sin x \, R(x) dx, \quad \int \cos x \, R(x) dx$$

may be treated in a similar manner. Either the integral is of the form

$$\cos x \, R_1(x) + \sin x \, R_2(x)$$

or it consists of a term of this kind together with a number of terms which involve the transcendents

$$\int \frac{\cos x}{x} dx, \qquad \int \frac{\sin x}{x} dx,$$

which are called the Cosine-integral and Sine-integral of x ($\operatorname{Ci} x$ and $\operatorname{Si} x$). These transcendents are of course not fundamentally distinct from the logarithm-integral.

5. Liouville has gone further and shown that it is always possible to determine whether the integral

$$\int (Pe^p + Qe^q + \dots + Te^t)\,dx,$$

where $P, Q, \dots, T, p, q, \dots, t$ are algebraical functions, is an elementary function, and to obtain the integral in case it is one*. The most general theorem which has been proved in this region of mathematics (also due to Liouville) is the following:

'*if $y, z, \dots$ are quantities whose differential coefficients are algebraical functions of $x, y, z, \dots$ and F denotes an algebraical function, and if*

$$\int F(x, y, z, \dots)\,dx$$

is expressible in finite terms, it is of the form

$$t + A\log u + B\log v + \dots$$

where $t, u, v, \dots$ are algebraical functions of $x, y, z, \dots$. For "algebraical" we may substitute "rational" throughout.'

Thus for example the theorem applies to

$$F(x, e^x, e^{e^x}, \log x, \log\log x, \cos x, \sin x)$$

since, if the various arguments of F are denoted by $x, y, z, \xi, \eta, \zeta, \theta$,

$$\frac{dy}{dx} = y, \qquad \frac{dz}{dx} = yz, \qquad \frac{d\xi}{dx} = \frac{1}{x},$$

$$\frac{d\eta}{dx} = \frac{1}{x\xi}, \qquad \frac{d\zeta}{dx} = -\sqrt{1-\zeta^2}, \qquad \frac{d\theta}{dx} = \sqrt{1-\theta^2}.$$

In spite of the immense generality of this theorem its proof is not particularly difficult, and does not involve ideas radically different from those which have been continually employed throughout the preceding pages.

6. As a final example of the manner in which these ideas may be applied we shall consider the following question:

'*under what circumstances is*

$$\int R(x)\log x\,dx$$

an elementary function, R being rational?'

In the first place the integral must be of the form

$$R_0(x, \log x) + A_1 \log R_1(x, \log x) + A_2 \log R_2(x, \log x) + \dots.$$

A general consideration of the form of the differential coefficient of this expression, in which $\log x$ must only occur linearly and multiplied by a rational function leads us to anticipate that (i) $R_0(x, \log x)$ is of the form

$$S(x)(\log x)^2 + T(x)\log x + W(x),$$

* An interesting particular result is that the 'error function' $\int e^{-x^2}dx$ is not an elementary function.

where S, T, and W are rational, and (ii) R_1, R_2,... must be rational functions of x only, so that the integral can be expressed in the form

$$S(x)(\log x)^2 + T(x)\log x + W(x) + \Sigma B_k \log(x - a_k).$$

Differentiating and comparing the result with the subject of integration we obtain the equations

$$S' = 0, \quad 2S/x + T' = R, \quad T/x + W' + \Sigma B_k/(x - a_k) = 0.$$

Hence S is a constant, say C, and

$$T = \int\left(R - \frac{2C}{x}\right) dx.$$

We can always determine by means of elementary operations (IV. 3) whether this integral is rational for any value of C or not.

If not the given integral cannot be expressed in finite form. If the integral is rational we calculate T. Then

$$W = -\int\left\{\frac{T}{x} + \Sigma \frac{B_k}{x - a_k}\right\} dx,$$

must be rational, for some value of the arbitrary constant implied in T. We can calculate the rational part of $\int \frac{T}{x} dx$: the transcendental part must be cancelled by the logarithmic terms

$$\Sigma B_k \log(x - a_k).$$

The necessary and sufficient condition that the integral should be an elementary function is therefore that R should be of the form

$$\frac{a}{x} + \frac{d}{dx}\{R_1(x)\},$$

where R_1 is rational. That the integral is in this case such a function becomes obvious if we integrate by parts, for

$$\int\left(\frac{a}{x} + R_1'\right)\log x\, dx = \tfrac{1}{2}a(\log x)^2 + R_1 \log x - \int \frac{R_1}{x} dx.$$

In particular

$$\text{(i)} \quad \int \frac{\log x}{x - a} dx, \qquad \text{(ii)} \quad \int \frac{\log x}{(x - a)(x - b)} dx,$$

are not elementary functions unless in (i) $a = 0$, and in (ii) $b = a$. If the integral is elementary the integration can always be carried out, with the same reservation as was necessary in the case of rational functions (IV. 5).

It is evident that the problem considered in this paragraph is but one of a whole class of similar problems. The reader will find it instructive to formulate and consider such problems for himself.

7. It will be obvious by now that the number of classes of transcendental functions whose integrals are always elementary is very small, and that such integrals as

$$\int f(x, e^x)\,dx, \qquad \int f(x, \log x)\,dx,$$
$$\int f(x, \cos x, \sin x)\,dx, \quad \int f(e^x, \cos x, \sin x)\,dx,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

where f is algebraical, or even rational, are generally new transcendents. These new transcendents, like the transcendents (such as the elliptic integrals) which arise from the integration of algebraical functions, are in many cases of great interest and importance. They may often be expressed by means of infinite series or definite integrals, or their properties may be studied by means of the integral expressions which define them. The very fact that such a function is *not* an elementary function in so far enhances its importance. And when such functions have been introduced into analysis new problems of integration arise in connection with them. We may enquire, for example, under what circumstances an elliptic integral or elliptic function, or a combination of such functions with elementary functions, can be integrated in finite terms by means of elementary and elliptic functions. But before we can be in a position to restate the fundamental problem of the Integral Calculus in any such more general form, it is essential that we should have disposed of the particular problem formulated in Section III.

APPENDIX.

The following is a complete list of the memoirs by Abel, Liouville and Tchebichef referred to in the text, which may be useful to the reader who wishes to pursue the subject further.

N. H. ABEL. 'Sur l'intégration de la formule différentielle $\frac{\rho dx}{\sqrt{R}}$.' *Crelle*, b. I.; *Œuvres*, t. I. p. 104.

—— 'Précis d'une Théorie des fonctions elliptiques.' *Crelle*, b. IV.; *Œuvres* t. I. p. 518 (esp. pp. 545 *et seq.*).

—— 'Théorie des transcendantes elliptiques,' (posthumous, unfinished). *Œuvres*, t. II. p. 87.

J. LIOUVILLE. 'Mémoire sur la classification des transcendantes.' *Journal de Math.*, Sér. I. t. II. p. 56.

—— 'Nouvelles recherches sur la détermination des intégrales dont la valeur est algébrique.' *Journal de Math.*, Sér. I. t. III. p. 22.

—— 'Suite du Mémoire sur la classification des transcendantes.' *Ibid.* p. 523.

—— 'Note sur les transcendantes elliptiques considérées comme fonctions de leur module.' *Journal de Math.*, Sér. I. t. V. p. 34 and p. 441.

—— 'Première Mémoire sur la détermination des intégrales dont la valeur est algébrique.' *Journal de l'École Polytechnique*, t. XIV. cah. 22, p. 124, and *Mémoires des Savants Étrangers*, t. V.

—— 'Seconde Mémoire.' *Ibid.* p. 149.

—— 'Mémoire sur les Transcendantes Elliptiques considérées comme fonctions de leur amplitude.' *Ibid.* cah. 23, p. 37.

—— 'Mémoire sur l'intégration d'une classe de fonctions transcendantes.' *Crelle*, b. XIII. p. 93.

P. TCHEBICHEF. 'Sur l'intégration des différentielles irrationnelles.' *Journal de Math.*, Sér. I. t. XVIII. p. 87.

—— 'Sur l'intégration des différentielles qui contiennent une racine carrée d'une polynome.' *Journal de Math.*, Sér. II. t. II. p. 1.

—— 'Sur l'intégration de la différentielle $(x+A)\,dx/\sqrt{x^4+\alpha x^3+\beta x^2+\gamma x+\delta}$.' *Journal de Math.*, Sér. II. t. II. p. 225.

—— 'Sur l'intégration des différentielles irrationnelles.' *Ibid.* p. 242.

—— 'Sur l'intégration des différentielles etc.' *Mémoires de l'Académie de St Pétersbourg*, Sér. VI. t. VI. p. 203.

Most of these memoirs relate to the integration of algebraical functions.

Reference may also be made to Hermite, *Cours d'Analyse*, t. I.; Laurent, *Traité d'Analyse*, t. IV.; Goursat, *Cours d'Analyse*, t. I.; Jordan, *Cours d'Analyse*, t. II.; Bertrand, *Calcul Intégral*, ch. V.; Wirtinger, *Encycl. d. Math. Wiss.*, II. B 2, § 36; Enneper, *Elliptische Functionen.*

The literature concerning pseudo-elliptic integrals and exceptional cases in Abelian integrals generally is very extensive. The following references may however be useful: Weierstrass, *Monatsberichte d. Ak. zu Berlin*, 1857; Raffy, *Annales de l'École Normale*, 1885; Zolotareff, *Journal de Math.* (2), t. XIX.; Greenhill, *Proc. Lond. Math. Soc.*, vol. XXV.; Dolbnia, *Journal de Math.* (4), t. VI. There are a number of papers by Dolbnia, Ptaszycki, and Kapteyn, in the *Bulletin des Sciences Math.*, and by Poincaré, Picard, Gunther, Raffy, and Goursat in the *Bulletin de la Soc. Math. de France.* Picard's *Traité d'Analyse* and Appell and Goursat's *Théorie des Fonctions Algebriques* should also be consulted.

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THE FUNDAMENTAL THEOREMS OF THE DIFFERENTIAL CALCULUS, by W. H. YOUNG, Sc.D.